DIMENSIONAL CROSSOVER AND EFFECTIVE EXPONENTS

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ABSTRACT

We investigate the critical behavior of the $\lambda\phi^4$ theory defined on $S^1 \times R^d$ having two finite length scales β, the circumference of S^1, and k^{-1}, the blocking scale introduced by the renormalization group transformation. By numerically solving the coupled differential RG equations for the finite-temperature blocked potential $U_{\beta,k}(\Phi)$ and the wavefunction renormalization constant $\mathcal{Z}_{\beta,k}(\Phi)$, we demonstrate how the finite-size scaling variable $\bar{\beta} = \beta k$ determines whether the phase transition is $(d+1)$- or d-dimensional in the limits $\bar{\beta} \gg 1$ and $\bar{\beta} \ll 1$, respectively. For the intermediate values of $\bar{\beta}$, finite-size effects play an important role. We also discuss the failure of the polynomial expansion of the effective potential near criticality.

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I. INTRODUCTION

The characteristics of a physical system often depend on the energy scale of interest as well as external environmental variables. Consider QCD as an example. At distances much less than the confinement scale $\Lambda_{\rm QCD}^{-1} \sim 1$ fm, asymptotic freedom gives a vanishing coupling strength between the quark and gluon fields. On the other hand, if the system is coupled to an external heat bath at temperature $T > \Lambda_{\rm QCD}$, a deconfinement transition takes place, leading to the formation of a quark-gluon plasma [1]. Although physics in the low energy or low T limit can in principle be accounted for by the fundamental quark and gluon degrees of freedom, it is more natural to use baryons and mesons as the effective degrees of freedom. This simple illustration shows how effective degrees of freedom can be altered by an environmental factor which in this case is T. The necessity of changing degrees of freedom in order to describe physical phenomena at different energy scales or environmental conditions is often indicative of crossover which is the interplay between two types of critical behaviors [2].

Another class of crossover of considerable interest is dimensional crossover, in which the system undergoes profound changes as its dimensionality, D, is varied. The dependence of the critical behavior on D is easily understood from the fact that D and the symmetry group of the order parameter are the essential elements for the classification of universality classes. Dimensional crossover can be studied by considering a system having a finite size in one or more dimensions. Our interest in dimensional crossover has its origins in the imaginary-time approach to finite-temperature field theory. Within this formalism the system is defined on the manifold $S^1 \times R^d$ with the inverse temperature $\beta = T^{-1}$ being the circumference of S^1, i.e., the system is infinite in d dimensions and finite in the remaining one. As the imaginary-time "thickness" β approaches infinity, one should recover the expected $T = 0$ result which is formally equivalent to R^{d+1}. In the high T limit, however, dimensional reduction takes place, yielding properties which are characteristic of R^d. Thus, decreasing β allows the system to crossover from $D = d + 1$ to d.

The use of finite-size scaling arguments [3] to explore dimensional crossover has been extensively discussed by O'Connor, Stephens, van Eijck and collaborators using the "Environmentally Friendly Renormalization" (EFR) prescription, and has had remarkable success in extracting universal quantities such as the critical exponents and the ratios of critical amplitudes [4]; similar approaches have also been presented in [5]. When the "environmental" or "anisotropic" factor T is such that $T \gg \mu_\beta$, where μ_β is the temperature-dependent mass of the theory, a new type of infrared (IR) divergence independent of the ultraviolet (UV) cutoff scale Λ appears. The idea of EFR is to construct a set of T-dependent counterterms so that not only UV singularities are subtracted off, but also the IR divergences associated with the ratio T/μ_β. The difficulty encountered in using T-independent counterterms stems from the fact that IR singularities are necessarily T-dependent. The conventional T-independent prescription fails to describe crossover since the degrees of freedom in the crossover regime differ from those near $T \sim 0$.

While EFR offers valuable insight into dimensional crossover, there exists an equally promising non-perturbative tool: the finite-temperature renormalization group (RG) [6]

[7]. This RG formulation is based on the Wilson-Kadanoff blocking transformation [8] and can be illustrated by considering the following scalar action

$$S[\phi] = \int dt \int_{\mathbf{x}} \Big\{ \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \Big\}, \qquad \int_{\mathbf{x}} = \int d^d\mathbf{x} \tag{1.1}$$

in d spatial plus one time dimensions. At finite-temperature the time dimension is compactified to S^1, and one applies the coarse-graining blocking transformation to the d-dimensional subsystem using an $O(d)$ symmetric smearing function $\rho_k^{(d)}(\mathbf{x})$. The original field, being periodic in S^1, can be expanded as

$$\phi(\mathbf{x},\tau_x) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau_x}\phi_n(\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int_{\mathbf{p}} e^{-i(\omega_n\tau_x - \mathbf{p}\cdot\mathbf{x})}\phi_n(\mathbf{p}), \qquad \int_{\mathbf{p}} = \int \frac{d^d\mathbf{p}}{(2\pi)^d}, \tag{1.2}$$

where $\omega_n = 2\pi n/\beta$ are the Matsubara frequencies. The resulting effective blocked field is then given by

$$\phi_k(\mathbf{x}) = \frac{1}{\beta}\int_0^\beta d\tau_y \int_{\mathbf{y}} \rho_k^{(d)}(\mathbf{x}-\mathbf{y})\phi(y) = \frac{1}{\beta}\int_{|\mathbf{p}|<k} e^{i\mathbf{p}\cdot\mathbf{x}}\phi_0(\mathbf{p}) = \Phi(\mathbf{x}), \tag{1.3}$$

where for simplicity, we have chosen

$$\rho_k^{(d)}(\mathbf{x}) = \int_{|\mathbf{p}|<k} e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{1.4}$$

or $\rho_k^{(d)}(\mathbf{p}) = \delta_{n,0}\Theta(k-|\mathbf{p}|)$, using $\int_0^\beta d\tau_x \exp(-i\omega_n\tau_x) = \beta\delta_{n,0}$. This can be contrasted with $\rho_k^{(d+1)}(p) = \Theta(k-p)$ constructed for R^{d+1} which corresponds to the zero-temperature limit of $S^1 \times R^d$ [9]. The advantage of making the smearing function a sharp cutoff in momentum space is that the scale k will provide a clear separation between the fast and slow modes. As can be seen from (1.3), our blocking in the imaginary-time formalism leads to an integration over all modes with $n \neq 0$ as well as the $|\mathbf{p}| > k$ modes for $n = 0$.

In order to probe the physics at an energy scale $E \sim k$, it is desirable to integrate out the irrelevant microscopic degrees of freedom between k and the UV cutoff Λ. The naive perturbative approach in which all the fast-fluctuating modes are integrated out at once can dramatically change the running parameters due to the incorporation of a large amount of fluctuations. For example, a significant shift in the mass parameter from its bare value μ_B^2 to $\tilde{\mu}_{\beta,k}^2$ is generally produced if $\Lambda \gg k$. Having large dressings means that the running of the parameters of the theory will not be tracked well by this type of RG trajectory [4]; poor tracking of the running parameters is why perturbation theory is plagued by severe IR singularities in the high-T limit and near the critical point. As an improvement, the Wegner-Houghton differential RG prescription [10] divides the integration volume into a large number of thin shells each having a thickness Δk. A systematic elimination of each shell is then performed until the desired scale is reached. In this manner, the continuous feedbacks from the higher modes to the lower ones are incorporated. Moreover, since the

running parameters are dressed infinitesimally along the RG trajectory, the effective degrees of freedom of the theory are accurately followed. This concept is of key importance in the formulation of the exact renormalization group [11]. In particular, such nonperturbative approach would remain applicable in the neighborhood of crossover where new effective degrees of freedom enter. Moreover, one can show that the IR divergences are completely lifted as the RG improved coupling constant tends to zero at criticality.

The arbitrariness of k, the IR cutoff, forms the basis for the momentum RG. By varying k infinitesimally from $k \to k - \Delta k$, we arrive at the following RG equation for the finite-temperature blocked potential $U_{\beta,k}(\Phi)$:

$$
\begin{aligned}
\dot{U}_{\beta,k} &= -\frac{S_d k^d}{\beta} \ln \sinh\Big(\frac{\beta\sqrt{k^2 + U''_{\beta,k}/\mathcal{Z}_{\beta,k}}}{2} \Big) \\
&= -\frac{S_d k^d}{2\beta} \Big\{ \beta\sqrt{k^2 + U''_{\beta,k}/\mathcal{Z}_{\beta,k}} + 2 \ln\Big[1 - e^{-\beta\sqrt{k^2 + U''_{\beta,k}/\mathcal{Z}_{\beta,k}}} \Big] \Big\},
\end{aligned}
\tag{1.5}
$$

where $S_d = 2/(4\pi)^{d/2}\Gamma(d/2)$ and the dot notation denotes the operation kd/dk. However, when no confusion arises, the same notation will be used for partial differentiation as well. The effect is the wavefunction renormalization constant $\mathcal{Z}_{\beta,k}$ is also incorporated. In (1.5), the first and the second terms may formally be interpreted as the contributions from quantum and thermal fluctuations, respectively, with the former being dominant in the low-T regime and the latter in the high-T limit. One may regard eq. (1.5) as the finite-temperature analog of the Wegner-Houghton differential RG equation [10].

From the example of QCD, it is quite clear that the effective degrees of freedom will vary from one regime to the other [12], and it would certainly not be suitable to attempt to use only one particular set of degrees of freedom to describe physical phenomena at all scales. In scalar field theory, we shall see that degrees of freedom characteristic of $(d+1)$ dimension can account for the low-T behavior of the system, and at sufficiently high T, the corresponding d-dimensional counterparts must be used. In the intermediate range, the system exhibits a mixture of the two limiting cases and is well described by the RG evolution of $U_{\beta,k}(\Phi)$. As we shall see later, such dimensional crossover is characterized by the dimensionless scale $\bar{\beta} = \beta k$.

In the present work, we follow closely the formalism developed in [6] and continue to explore the role of $U_{\beta,k}(\Phi)$ in describing physical phenomena at various temperature and momentum scales. As an extension of our previous work, we couple to the evolution of $U_{\beta,k}(\Phi)$ an additional RG equation arising from the consideration of $\mathcal{Z}_{\beta,k}(\Phi)$. Solutions to the two coupled nonlinear partial differential equations will provide a smooth connection between the small- and large distance physics at arbitrary finite temperature. In particular, the way in which $\mathcal{Z}_{\beta,k}(\Phi)$ influences the critical behavior of the system can be seen in a rather transparent manner. We shall demonstrate that our RG approach is equally as "environmentally friendly" as that advocated in [4].

The organization of the paper is as follows: In Sec. II the perturbative one-loop finite-temperature blocked potential $\tilde{U}_{\beta,k}(\Phi)$ and the wavefunction renormalization constant $\tilde{Z}_{\beta,k}(\Phi)$ for one-component scalar theory are derived based on the momentum blocking method. In Sec. III we use the results of Sec. II and generate the coupled RG flow equations

for the improved $U_{\beta,k}(\Phi)$ and $\mathcal{Z}_{\beta,k}(\Phi)$. In addition to solving these equations explicitly, we also approximate $U_{\beta,k}(\Phi)$ by polynomial expansion. Sec. IV contains the discussions of dimensional crossover. It is shown that the phenomenon takes place approximately at the scale $\bar{\beta} \sim 1$. If the system is initially in a broken phase, at sufficiently high T, we expect the symmetry to be restored. Numerical calcualtions of the critical exponents are presented in Sec. V for D equal to three and four by considering the limits $\bar{\beta} \gg 1$ and $\bar{\beta} \ll 1$, respectively, and are seen to be in excellent agreement with previous results. Sec. VI is reserved for summary and discussions.

II. FINITE-TEMPERATURE SCALAR THEORY

We first consider the following bare action

$$S[\phi] = \int_0^\beta d\tau \int_{\mathbf{x}} \left\{ \frac{Z}{2}(\partial_\tau \phi)^2 + \frac{Z}{2}(\nabla\phi)^2 + V(\phi) \right\}, \tag{2.1}$$

defined on the manifold $S^1 \times R^d$, with Z being the bare wavefunction renormalization constant which conventionally is taken to be unity. The action corresponds to a $(d+1)$-dimensional layered classical system of thickness β or a d-dimensional quantum system of "time" extent $\beta\hbar$. Via the coarse-graining blocking transformation, a new blocked action $\widetilde{S}_{\beta,k}[\Phi]$ which can be obtained [7]:

$$e^{-\widetilde{S}_{\beta,k}[\Phi(\mathbf{x})]} = \int_{\text{periodic}} D[\phi] \prod_{\mathbf{x}} \delta(\phi_k(\mathbf{x}) - \Phi(\mathbf{x})) e^{-S[\phi]}, \tag{2.2}$$

where the field average of a given block $\Phi(\mathbf{x})$ is chosen to coincide with the slowly varying background since $\phi_k(\mathbf{p}) = \rho_k^{(d)}(\mathbf{p})\phi_n(\mathbf{p}) = \Theta(k - |\mathbf{p}|)\phi_0(\mathbf{p})$ with (1.4). An alternative choice for the smearing function is

$$\rho_{\tilde{n},k}^{(d)}(\mathbf{x}, \tau_x) = \rho_{\tilde{n}}(\tau_x)\rho_k^{(d)}(\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\tilde{n}}^{\tilde{n}} \int_{|\mathbf{p}|<k} e^{-i(\omega_n \tau_x - \mathbf{p}\cdot\mathbf{x})}, \tag{2.3}$$

which gives

$$\begin{aligned}
\phi_k(\mathbf{x}, \tau_x) &= \frac{1}{\beta}\int_0^\beta d\tau_y \int_{\mathbf{y}} \rho_{\tilde{n},k}^{(d)}(\mathbf{x}-\mathbf{y}, \tau_x - \tau_y)\phi(y) = \frac{1}{\beta}\sum_{n=-\tilde{n}}^{\tilde{n}} \int_{|\mathbf{p}|<k} e^{-i(\omega_n \tau_x - \mathbf{p}\cdot\mathbf{x})}\phi_n(\mathbf{p}) \\
&= \frac{1}{\beta}\int_{|\mathbf{p}|<k} e^{i\mathbf{p}\cdot\mathbf{x}} \Big\{ \phi_0(\mathbf{p}) + e^{-i\omega_1\tau_x}\phi_1(\mathbf{p}) + e^{i\omega_1\tau_x}\phi_{-1}(\mathbf{p}) + \cdots \\
&\qquad + e^{-i\omega_{\tilde{n}}\tau_x}\phi_{\tilde{n}}(\mathbf{p}) + e^{i\omega_{\tilde{n}}\tau_x}\phi_{-\tilde{n}}(\mathbf{p}) \Big\} = \Phi(\mathbf{x}, \tau_x).
\end{aligned} \tag{2.4}$$

The difference between (1.4) and (2.3) is that in the latter, in addition to truncating the higher momentum modes by the step function $\Theta(k-|\mathbf{p}|)$, bounds ($\pm\tilde{n}$) have been placed on the summation over the Matsubara frequencies with $\rho_{\tilde{n}}(\tau_x)$. Thus, $\Phi(\mathbf{x},\tau_x)$ differs from $\Phi(\mathbf{x})$ in that it contains contributions from all ω_n with $n\le\tilde{n}$. Nevertheless, since in the high T regime all except for the static $n=0$ modes are strongly suppressed, it suffices to use (1.3) knowing that modes with $n\neq 0$ can be treated perturbatively. Notice that the form of $\rho^{(d)}_{\tilde{n},k}(\mathbf{x},\tau_x)$ is not unique; any prescription that incorporates the $n=0$ mode with $|\mathbf{p}|<k$ is equally suitable.

The one-loop contribution which takes into account the quadratic order of the fluctuations can be written as

$$\tilde{S}^{(1)}_{\beta,k}[\Phi]=\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int_{\mathbf{x}}\int_{\mathbf{p}}^{\prime}\ln\Big[Z\omega_n^2+Z\mathbf{p}^2+V''(\Phi)\Big],\qquad \int_{\mathbf{p}}^{\prime}=S_d\int_k^{\Lambda}dp\,p^{d-1},\tag{2.5}$$

where $p=|\mathbf{p}|$ and we have made the substitutions

$$p_0\longrightarrow\omega_n=\frac{2\pi n}{\beta},\qquad \int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\longrightarrow\frac{1}{\beta}\sum_{n=-\infty}^{\infty},\tag{2.6}$$

in going to the imaginary-time formalism. We denote quantities computed using the perturbative approach with a small tilde, to be distinguished from the RG results in the later sections. At low T where the gap between the adjacent modes becomes small, the summation over n can be replaced by an integration and one readily recovers the $(d+1)$-dimensional classical theory.

In the low-energy limit, derivative expansion can be employed. Following the method by Fraser [13], we write the background field as $\Phi(\mathbf{x})=\Phi_0+\tilde{\Phi}(\mathbf{x})$ where $\tilde{\Phi}(\mathbf{x})$ represents the small spatial inhomogeneities that can be treated perturbatively. Derivative terms are then produced by treating $\tilde{\Phi}(\mathbf{x})$ and $\mathbf{p}$ as operators obeying the following commutation relations:

$$[p_i,\tilde{\Phi}]=i\partial_i\tilde{\Phi},\qquad [\mathbf{p}^2,\tilde{\Phi}]=2ip_i\partial_i\tilde{\Phi}+\nabla^2\tilde{\Phi}.\tag{2.7}$$

After carrying out the above procedures, (2.5) becomes

$$\begin{aligned}\tilde{S}^{(1)}_{\beta,k}[\Phi]&=\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int_x\int_{\mathbf{p}}^{\prime}\Bigg\{\ln\big[Z\omega_n^2+Z\mathbf{p}^2+V''(\Phi_0)\big]+\frac{V'''(\Phi_0)\tilde{\Phi}+\frac{1}{2}V''''(\Phi_0)\tilde{\Phi}^2}{Z\omega_n^2+Z\mathbf{p}^2+V''(\Phi_0)}\\&-\frac{1}{2}\Big[\frac{1}{Z\omega_n^2+Z\mathbf{p}^2+V''(\Phi_0)}V'''(\Phi_0)\tilde{\Phi}\frac{1}{Z\omega_n^2+Z\mathbf{p}^2+V''(\Phi_0)}V'''(\Phi_0)\tilde{\Phi}\Big]\Bigg\}+O(\tilde{\Phi}^3)\\&\equiv\int_x\Bigg\{-\frac{\tilde{Z}^{(1)}_{\beta,k}(\Phi_0)}{2}\tilde{\Phi}\nabla^2\tilde{\Phi}+\tilde{U}^{(1)}_{\beta,k}(\Phi_0)+\tilde{U}^{(1)'}_{\beta,k}(\Phi_0)\tilde{\Phi}+\frac{1}{2}\tilde{U}^{(1)''}_{\beta,k}(\Phi_0)\tilde{\Phi}^2+\cdots\Bigg\},\end{aligned}\tag{2.8}$$

where $\tilde{Z}^{(1)}_{\beta,k}(\Phi_0)$ is the one-loop correction to the wave function renormalization constant. Upon replacing Φ_0 by $\Phi(\mathbf{x})$ in (2.8) [13], we are led to

$$\begin{aligned}\tilde{U}^{(1)}_{\beta,k}(\Phi) &= \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int_{\mathbf{p}}^{\prime} \ln\Big[Z\omega_n^2 + Z\mathbf{p}^2 + V''(\Phi)\Big] \\ &= \frac{1}{2\beta}\int_{\mathbf{p}}^{\prime}\left\{\beta\sqrt{\mathbf{p}^2+\tilde{V}''} + 2\ln\Big[1-e^{-\beta\sqrt{\mathbf{p}^2+\tilde{V}''}}\Big]\right\} + \cdots,\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\tilde{Z}^{(1)}_{\beta,k}(\Phi) &= \frac{(\tilde{V}''')^2}{2\beta}\sum_{n=-\infty}^{\infty}\int_{\mathbf{p}}^{\prime}\frac{-\mathbf{p}^2/3+\omega_n^2+\tilde{V}''}{(\omega_n^2+\mathbf{p}^2+\tilde{V}'')^4} \\ &= \frac{(\tilde{V}''')^2 S_d}{36\beta}\int_0^{\infty} dss^3\int_k^{\infty} dpp^{d-1}e^{-(p^2+\tilde{V}'')s}\sum_{n=-\infty}^{\infty}\big[-p^2+3\omega_n^2+3\tilde{V}''\big]e^{-\omega_n^2 s}.\end{aligned} \tag{2.10}$$

where $\tilde{V}^{(n)} = V^{(n)}/Z$. Performing the summation using the Poisson formulae then gives

$$\begin{aligned}\tilde{Z}^{(1)}_{\beta,k}(\Phi) &= \frac{(\tilde{V}''')^2 S_d}{72\pi^{1/2}}\int_0^{\infty} ds\ s^{3/2}\int_k^{\infty} dpp^{d-1}e^{-(p^2+\tilde{V}'')s} \\ &\times\left\{\frac{3}{2} + (-p^2+3\tilde{V}'')s + 2\sum_{n=1}^{\infty}\Big[\frac{3}{2}(1-\frac{\beta^2n^2}{2s}) + (-p^2+3\tilde{V}'')s\Big]e^{-\beta^2n^2/4s}\right\}.\end{aligned} \tag{2.11}$$

The first two terms inside the braces of the above expression constitute the usual zero-temperature correction. Eqs. (2.9) and (2.11) correspond to the standard one-loop calculation with derivative expansion and are valid in the small momentum limit where the effective blocked action $\widetilde{S}_{\beta,k}[\Phi]$ can be characterized by $\tilde{U}_{\beta,k}(\Phi)$, $\tilde{Z}_{\beta,k}(\Phi)$ and other higher-order derivative terms. However, the presence of IR singularities in the high T regime will invalidate perturbation theory, and the situation can be remedied only if the dominant higher loop corrections are also resummed. The goal of achieving resummation via RG methods is the subject of the next section.

III. RENORMALIZATION GROUP IMPROVEMENT

Within our framework of finite-temperature RG, there exists two arbitrary length parameters, k^{-1} and β whose ratio defines the dimensionless quantity $\bar{\beta} = \beta k$. The momentum scale k naturally enters as the effective IR cutoff for the R^d subsystem during the course of blocking transformation. Thus, it is most desirable for the coarse-graining procedure which employs blocks of size βk^{-d} as the effective degrees of freedom to be continued until k^{-1}, the linear dimension of the blocks becomes comparable to the characteristic correlation length of the fluctuations in the system, i.e., $k^{-1} \sim \xi$. By doing so, the critical behavior of the entire system can be probed by simply examining the characteristic behavior of any representative block.

To set up the RG formalism, we first differentiate (2.9) with respect to the arbitrary IR scale k:

$$\dot{\tilde{U}}_{\beta,k} = -\frac{S_d k^d}{2\beta}\left\{\beta\sqrt{k^2+\tilde{V}''} + 2\ln\left[1 - e^{-\beta\sqrt{k^2+\tilde{V}''}}\right]\right\}. \tag{3.1}$$

Similarly, after summing over the Matsubara frequencies in (2.10), one obtains

$$\begin{aligned}\dot{\tilde{Z}}_{\beta,k} = -\frac{(\tilde{V}''')^2 S_d k^d}{48\big(k^2+\tilde{V}''\big)^{7/2}}\Bigg\{&\big(-k^2+9\tilde{V}''\big)\left[\frac{1}{2} + n_k\big(1+\beta(k^2+\tilde{V}'')^{1/2}(1+n_k)\big)\right] \\ &+ n_k(1+n_k)\beta^2(k^2+\tilde{V}'')^2\Big[\big(-k^2+3\tilde{V}''\big)(1+2n_k) \\ &- \frac{2}{3}k^2\beta(k^2+\tilde{V}'')^{1/2}(1+6n_k+6n_k^2)\Big]\Bigg\},\end{aligned} \tag{3.2}$$

where $n_k = \big(e^{-\beta\sqrt{k^2+\tilde{V}''}} - 1\big)^{-1}$.

The above two differential equations are obtained by integrating out each mode independently, ignoring the feedback from fast modes to slow modes as the IR cutoff is lowered. Such an "independent-mode approximation" (IMA) only accounts for contributions up to the one-loop order. In our RG approach, instead of integrating out all the modes between Λ and k at once, one first divides the integration volume into a large number of thin shells of small thickness Δk, and lower the cutoff infinitesimally from $\Lambda \to \Lambda - \Delta k$ until $\Lambda = k$ is reached. In this manner, we arrive at the following coupled RG equations:

$$\left[k\partial_k - \frac{1}{2}(d-2+\eta)\bar{\Phi}\partial_{\bar{\Phi}} + d\right]\bar{U}_{\beta,k} = -\frac{S_d}{2}\left\{\bar{\beta}\sqrt{1+\hat{U}''_{\beta,k}} + 2\ln\left[1 - e^{-\bar{\beta}\sqrt{1+\hat{U}''_{\beta,k}}}\right]\right\}, \tag{3.3}$$

and

$$\begin{aligned}&\big(k\partial_k - \eta\big)\bar{\mathcal{Z}}_{\beta,k} \\ &= -\frac{(\hat{U}'''_{\beta,k})^2\bar{\beta}S_d}{48\big(1+\hat{U}''_{\beta,k}\big)^{7/2}}\Bigg\{\big(-1+9\hat{U}''_{\beta,k}\big)\left[\frac{1}{2} + n_{\bar{\beta}}\Big\{1+\bar{\beta}\big(1+\hat{U}''_{\beta,k}\big)^{1/2}(1+n_{\bar{\beta}})\Big\}\right] \\ &+ n_{\bar{\beta}}(1+n_{\bar{\beta}})\bar{\beta}^2(1+\hat{U}''_{\beta,k})\Big[(-1+3\hat{U}''_{\beta,k})(1+2n_{\bar{\beta}}) - \frac{2\bar{\beta}}{3}\big(1+\hat{U}''_{\beta,k}\big)^{1/2}\big(1+6n_{\bar{\beta}}+6n_{\bar{\beta}}^2\big)\Big]\Bigg\},\end{aligned} \tag{3.4}$$

where the dimensionless quantities are defined as

$$\begin{aligned}&\bar{U}_{\beta,k}(\bar{\Phi}) = \beta k^{-d}U_{\beta,k}(\Phi), \qquad \bar{\Phi} = \beta^{1/2}k^{-(d-2+\eta)/2}\Phi, \qquad \bar{\beta} = \beta k, \\ &\bar{U}^{(m)}_{\beta,k}(\bar{\Phi}) = \frac{\partial^m \bar{U}_{\beta,k}(\bar{\Phi})}{\partial\bar{\Phi}^m} = \beta^{1-m/2}k^{-d+m(d-2+\eta)/2}U^{(m)}_{\beta,k}(\Phi), \\ &\bar{\mu}^2_{\beta,k} = \bar{U}^{(2)}_{\beta,k}(0) = k^{-2+\eta}\mu^2_{\beta,k}, \quad \bar{\lambda}_{\beta,k} = \bar{U}^{(4)}_{\beta,k}(0) = \beta^{-1}k^{d-4+2\eta}\lambda_{\beta,k}, \\ &\bar{\mathcal{Z}}_{\beta,k} = k^{\eta}\mathcal{Z}_{\beta,k}, \qquad \hat{U}^{(n)}_{\beta,k}(\bar{\Phi}) = \bar{U}^{(n)}_{\beta,k}(\bar{\Phi})/\bar{\mathcal{Z}}_{\beta,k}(\bar{\Phi}), \\ &n_{\bar{\beta}} = \big(e^{\bar{\beta}\sqrt{1+\hat{U}''_{\beta,k}}} - 1\big)^{-1}.\end{aligned} \tag{3.5}$$

These two coupled nonlinear partial differential equations provide a smooth connection between the small- and large-distance physics at finite temperature. The power of (3.3) and (3.4) is that they systematically incorporate the contributions of a particular mode for the elimination of the next, thereby taking into consideration not only the daisy and superdaisy graphs, but also all higher-order nonoverlapping loop diagrams. Notice that in analogy to (1.5), the first term inside the braces of (3.3) represents the effect of quantum fluctuations and dominates at low T, and contributions from thermal fluctuations appearing in the second term become dominant at large T. In Fig. 1 we illustrate the temperature dependence of $\mathcal{Z}_{\beta,k=0}(\Phi)$. In a similar manner, the evolution equation for the m-point scale-dependent vertex function $\bar{U}^{(m)}_{\beta,k}(\bar{\Phi})$ can be written as

$$\left[k\partial_k - \frac{1}{2}\big(d-2+\eta\big)\big(m+\bar{\Phi}\partial_{\bar{\Phi}}\big)+d\right]\bar{U}^{(m)}_{\beta,k}(\bar{\Phi}) = -S_d\partial^m_{\bar{\Phi}}\left[\ln\sinh\Big(\frac{\bar{\beta}\sqrt{1+\hat{U}''_{\beta,k}(\bar{\Phi})}}{2}\Big)\right], \quad (3.6)$$

upon differentiating (3.3) with respect to $\bar{\Phi}$.

One may compare (3.3) with the corresponding zero-temperature d-dimensional flow equation for the blocked potential $\bar{U}_{k,d}(\bar{\Phi}_d)$ [9]:

$$\left[k\partial_k - \frac{1}{2}\big(d-2+\eta\big)\bar{\Phi}_d\partial_{\bar{\Phi}_d}+d\right]\bar{U}_{k,d}(\bar{\Phi}_d) = -\frac{S_d}{2}\ln\Big[1+\hat{U}''_{k,d}(\bar{\Phi}_d)\Big], \quad (3.7)$$

where

$$\bar{U}_{k,d}(\bar{\Phi}_d)=k^{-d}U_{k,d}(\Phi_d), \qquad \bar{\mathcal{Z}}^{-1}_{k,d}(\bar{\Phi}_d)=k^{\eta}\mathcal{Z}^{-1}_{k,d}(\Phi_d), \qquad \bar{\Phi}_d=k^{-(d-2+\eta)/2}\Phi_d,$$
$$\hat{U}''_{k,d}=\partial^2_{\bar{\Phi}_d}\bar{U}_{k,d}/\bar{\mathcal{Z}}_{k,d}, \quad \bar{g}^{(m)}_{k,d}=\bar{U}^{(m)}_{k,d}(0)=k^{-d+m(d-2+\eta)/2}g^{(m)}_{k,d}, \qquad \bar{\lambda}_{k,d}=k^{d-4+2\eta}\lambda_{k,d}. \quad (3.8)$$

The major difference between these two equations is that the RG flow at finite temperature depends on the additional dimensionless parameter $\bar{\beta}$. However, under extreme temperature conditions, the RG evolution equation can be greatly simplified. Consider first the high T limit where $\bar{\beta}\ll 1$. The main contribution to the flow comes primarily from thermal fluctuations, the second term in the braces of (3.3). Neglecting the effect of quantum fluctuations entirely allows us to write

$$\begin{aligned}\left[k\partial_k - \frac{1}{2}\big(d-2+\eta\big)\bar{\Phi}_d\partial_{\bar{\Phi}_d}+d\right]\bar{U}_{k,d}(\bar{\Phi}_d) &= -S_d\ln\left[\frac{\bar{\beta}\sqrt{1+\hat{U}''_{k,d}(\bar{\Phi}_d)}}{\bar{\beta}\sqrt{1+\hat{U}''_{k,d}(0)}}\right]\\ &= -\frac{S_d}{2}\ln\left[\frac{1+\hat{U}''_{k,d}(\bar{\Phi}_d)}{1+\hat{U}''_{k,d}(0)}\right],\end{aligned} \quad (3.9)$$

which apart from an irrelevant normalization factor is completely equivalent to (3.7). The scaling relations between dimensionful and dimensionless quantities in this region are then

given by

$$\bar\beta \longrightarrow 0: \quad \begin{cases} S^1\times R^d \longrightarrow R^d & \big(D\longrightarrow d\big) \\ \bar\Phi \longrightarrow \bar\Phi_d & \big(\Phi \longrightarrow \beta^{-1/2}\Phi_d\big) \\ \bar U_{\beta,k} \longrightarrow \bar U_{k,d} & \big(U_{\beta,k}\longrightarrow \beta^{-1}U_{k,d}\big) \\ \bar{\mathcal Z}_{\beta,k} \longrightarrow \bar{\mathcal Z}_{k,d} & \big(\mathcal Z_{\beta,k}\longrightarrow \mathcal Z_{k,d}\big) \\ \bar g^{(m)}_{\beta,k} \longrightarrow \bar g^{(m)}_{k,d} & \big(g^{(m)}_{\beta,k}\longrightarrow \beta^{m/2-1}g^{(m)}_{k,d}\big). \end{cases} \tag{3.10}$$

Thus, we see that in the high-T limit, the system is reduced to a d-dimensional classical theory at zero T [14]. On the other hand, in the low T regime where $\bar\beta(1+\hat U''_{\beta,k}(\bar\Phi))^{1/2}\gg 1$, quantum fluctuations prevail and give

$$\Big[k\partial_k - \frac{1}{2}\big(d-2+\eta\big)\bar\Phi\partial_{\bar\Phi} + d\Big]\bar U_{k,d+1} = -\frac{S_d}{2}\sqrt{1+\hat U''_{k,d+1}}, \tag{3.11}$$

upon neglecting the thermal contributions. The $(d+1)$-dimensional characteristic of the system in this limit suggests the following scaling relations:

$$\bar\beta \longrightarrow \infty: \quad \begin{cases} S^1\times R^d \longrightarrow R^{d+1} & \big(D\to d+1\big) \\ \bar\Phi \longrightarrow \bar\beta^{1/2}\bar\Phi_{d+1} & \big(\Phi \longrightarrow \Phi_{d+1}\big) \\ \bar U_{\beta,k} \longrightarrow \bar\beta\bar U_{k,d+1} & \big(U_{\beta,k}\longrightarrow U_{k,d+1}\big) \\ \bar{\mathcal Z}_{\beta,k} \longrightarrow \bar{\mathcal Z}_{k,d+1} & \big(\mathcal Z_{\beta,k}\longrightarrow \mathcal Z_{k,d+1}\big) \\ \bar g^{(m)}_{\beta,k} \longrightarrow \bar\beta^{1-m/2}\bar g^{(m)}_{k,d+1} & \big(g^{(m)}_{\beta,k}\longrightarrow g^{(m)}_{k,d+1}\big). \end{cases} \tag{3.12}$$

In order to see the equivalence between (3.11) and the RG equation that possesses $O(d+1)$ symmetry, we make use of the following transformation

$$S_d\sqrt{1+\hat U''_{k,d+1}} \longrightarrow S_{d+1}\ln\Big[1+\hat U''_{k,d+1}\Big], \tag{3.13}$$

which is obtained from [6]

$$\begin{aligned} S_d\int_0^k d\mathbf{p}\mathbf{p}^{d-1}\sqrt{\mathbf{p}^2+U''} &= \int^k \frac{d^d\mathbf{p}}{(2\pi)^{d/2}}\sqrt{\mathbf{p}^2+U''} = \int^k \frac{d^d\mathbf{p}}{(2\pi)^{d/2}}\int\frac{dp_0}{2\pi}\ln(p_0^2+\mathbf{p}^2+U'') \\ = \frac{2\Omega_d}{(2\pi)^{d+1}}\int_0^{\pi/2}d\theta\cos^{d-1}\theta\int_0^{\sqrt2 k}dp\; p^d\ln\big(p^2+U''\big) &= S_{d+1}\int_0^{\sqrt2 k}dp\; p^d\ln\big(p^2+U''\big), \end{aligned} \tag{3.14}$$

with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ being the d-dimensional solid angle, and

$$\begin{aligned} k\frac{d}{dk} &= k\partial_k + k\Big(\frac{\partial\bar\Phi_d}{\partial k}\Big)\partial_{\bar\Phi_d} = k\partial_k - \frac{1}{2}\big(d-2+\eta\big)\bar\Phi_d\partial_{\bar\Phi_d} \\ &= k\partial_k + k\Big(\frac{\partial\bar\Phi_{d+1}}{\partial k}\Big)\partial_{\bar\Phi_{d+1}} = k\partial_k - \frac{1}{2}\big(d-1+\eta\big)\bar\Phi_{d+1}\partial_{\bar\Phi_{d+1}}. \end{aligned} \tag{3.15}$$

Thus, the $O(d)$-symmetric (3.11) can be equated with (3.7), with d replaced by $d+1$. The above scaling patterns illustrate the role played by the parameter $\bar{\beta}$ in the crossover from $(d+1)-$ to d-dimensional and vice versa. Moreover, dimensional crossover makes no reference to the specific structure of $\bar{U}_{\beta,k}(\bar{\Phi})$; it only requires that the condition $\bar{\beta}(1+\partial^2_{\bar{\Phi}}\bar{U}_{\beta,k}(\bar{\Phi}))^{1/2} \ll 1$ be fulfilled. The manner in which $\bar{\beta}$ affects D may be understood from the following heuristic argument: The representative degrees of freedom after applying the blocking transformation are blocks of finite volume βk^{-d} which becomes infinite as $k \to 0$. The boundary effect of S^1 can be neglected when the linear dimension of the block falls below β and the manifold $S^1 \times R^d$ is equivalent to R^{d+1} for all practical purpose. That is, for $k \to 0$ such that $\bar{\beta} \gg 1$, the block volume becomes "$\infty \times \infty^d = \infty^{d+1}$." Conversely, when k^{-1} is much larger than β such that $\bar{\beta} \ll 1$, the submanifold S^1 becomes "unnoticed" and the system undergoes a crossover to R^d (block volume $\sim \beta \times \infty^d \sim \infty^d$). Thus, the dimensional crossover scale can be established as $\bar{\beta} \sim 1$.

It is also important to point out that the reduction of D from $d+1$ to d in the limit $\bar{\beta} \to 0$ can be realized in two distinctive physical situations. The first possibility is to hold β fixed while having $k \to 0$. So long as the transition does not take place at $T_c = 0$, the critical behavior will always be controlled by the d-dimensional fixed point. The other possibility, sending β to zero while keeping k finite, coincides with the well-known scenario of high-T dimensional reduction. Thus, we see that it is $\bar{\beta}$, and not the magnitude of β itself, that provides the indication of whether the system is in the "high" or "low" temperature limit. Notice that the latter case applies irrespective of whether the vacuum symmetry is spontaneously broken or not.

B. Polynomial Expansion of $U_{\beta,k}(\Phi)$

The nonlinearity of the flow equations (3.3) and (3.4) makes analytical solutions rather difficult. While numerical solutions can be generated, very little is known concerning the nature of the operators involved. The simplest possible approximation is to write the potential as

$$\bar{U}_{\beta,k}(\bar{\Phi}) = \sum_{m=1}^{\infty} \frac{\bar{g}^{(2m)}_{\beta,k}}{(2m)!}\bar{\Phi}^{2m}, \qquad \bar{g}^{(2m)}_{\beta,k} = \bar{U}^{(2m)}_{\beta,k}(0), \tag{3.16}$$

where $\bar{g}^{(2)}_{\beta,k}$ and $\bar{g}^{(4)}_{\beta,k}$ can be identified as the k-dependent finite-temperature mass parameter $\bar{\mu}^2_{\beta,k}$ and coupling constant $\bar{\lambda}_{\beta,k}$, respectively. A truncation of the Taylor series expansion can be made by assuming the smallness of the higher order terms. Thus, keeping only the leading order contributions, one could then approximate the theory by the following coupled flow equations:

$$\Big[k\partial_k + 2 - \eta\Big]\bar{\mu}^2_{\beta,k} = -\frac{\bar{\lambda}_{\beta,k}S_d\bar{\beta}}{4\bar{u}_{\bar{\beta}}}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big), \tag{3.17}$$

$$\begin{aligned}\Big[k\partial_k + \big(4-d-2\eta\big)\Big]\bar{\lambda}_{\beta,k} = {} & \frac{3\bar{\lambda}^2_{\beta,k}S_d\bar{\beta}^2}{8\bar{u}^2_{\bar{\beta}}}\Big\{\frac{1}{\bar{\beta}\bar{u}_{\bar{\beta}}}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big) + \frac{1}{2}\mathrm{csch}^2\Big(\frac{\bar{\beta}\bar{u}_{\beta}}{2}\Big)\Big\} \\ & - \frac{\bar{g}^{(6)}_{\beta,k}S_d\bar{\beta}}{4\bar{u}_{\bar{\beta}}}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big),\end{aligned} \tag{3.18}$$

and

$$\Big[k\partial_k + \big(6-2d-3\eta\big)\Big]\bar{g}^{(6)}_{\beta,k} = -\frac{45\bar{\lambda}^3_{\beta,k}S_d\bar{\beta}}{16\bar{u}^3_{\bar{\beta}}}\Big\{\frac{\bar{\beta}^2}{6}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big)\mathrm{csch}^2\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big) + \frac{1}{\bar{u}^2_{\bar{\beta}}}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big) + \frac{\bar{\beta}}{2\bar{u}_{\bar{\beta}}}\mathrm{csch}^2\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big)\Big\} + \frac{15\bar{\lambda}_{\beta,k}\bar{g}^{(6)}_{\beta,k}S_d\bar{\beta}}{8\bar{u}^2_{\bar{\beta}}}\Big\{\frac{1}{\bar{u}_{\bar{\beta}}}\coth\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big) + \frac{\bar{\beta}}{2}\mathrm{csch}^2\Big(\frac{\bar{\beta}\bar{u}_{\bar{\beta}}}{2}\Big)\Big\} + O(\bar{g}^{(8)}_{\beta,k}), \tag{3.19}$$

where $\bar{u}_{\bar{\beta}} = (1+\bar{\mu}^2_{\beta,k})^{1/2}$. In a similar manner, the flow of $\bar{\mathcal{Z}}_{\beta,k}(\bar{\Phi})$ can be taken into account by substituting (3.16) into (3.4). In Sec. V we compare the numerical results for the critical exponents obtained from keeping the polynomial series up to $\bar{\Phi}^{10}$ with that generated by solving (3.3) and (3.4) directly without resorting to any approximation. What we shall find is that the former first converges to the latter, but then ceases to do so beyond a certain order [15]. The nonconverging behavior can be understood from the presence of singularities in the vertex functions $g^{(2m)}_{\beta,k}$ for $m \geq 4$, as we shall see.

When the symmetry is spontaneously broken at $T=0$ or when k becomes very large, the potential will have a nontrivial k- and β-dependent minimum $\hat{\bar{\Phi}}_{\beta,k}$. By assuming the potential to be of the form (3.16), one my locate $\hat{\bar{\Phi}}_{\beta,k}$ by solving

$$0 = \frac{\partial \bar{U}_{\beta,k}}{\partial\bar{\Phi}}\Big|_{\hat{\bar{\Phi}}_{\beta,k}} = \sum_{m=1}^{\infty}\frac{\bar{g}^{(2m)}_{\beta,k}}{(2m-1)!}\hat{\bar{\Phi}}^{2m-2}_{\beta,k}. \tag{3.20}$$

Differentiating (3.20) with respect to ln$k$ then yields

$$0 = \dot{\bar{U}}'_{\beta,k}(\hat{\bar{\Phi}}_{\beta,k}) = \sum_{m=1}^{\infty}\frac{\hat{\bar{\Phi}}^{(2m-3)}_{\beta,k}}{(2m-1)!}\Big\{\dot{\bar{g}}^{(2m)}_{\beta,k}\hat{\bar{\Phi}}_{\beta,k} + (2m-2)\bar{g}^{(2m)}_{\beta,k}\dot{\hat{\bar{\Phi}}}_{\beta,k}\Big\}, \tag{3.21}$$

or

$$\dot{\hat{\bar{\Phi}}}_{\beta,k} = -\frac{\sum_{m=1}^{\infty}\frac{\dot{\bar{g}}^{(2m)}_{\beta,k}}{(2m-1)!}\hat{\bar{\Phi}}^{2m-2}_{\beta,k}}{\sum_{m=1}^{\infty}\frac{\bar{g}^{(2m)}_{\beta,k}}{(2m-2)!}\hat{\bar{\Phi}}^{2m-3}_{\beta,k}} = -\frac{\dot{\bar{\mu}}^2{}_{\beta,k} + \frac{\dot{\bar{\lambda}}_{\beta,k}}{3!}\hat{\bar{\Phi}}^2_{\beta,k} + \frac{\dot{\bar{g}}^{(6)}_{\beta,k}}{5!}\hat{\bar{\Phi}}^4_{\beta,k} + \cdots}{\frac{2\bar{\lambda}_{\beta,k}}{3!}\hat{\bar{\Phi}}_{\beta,k} + \frac{4\bar{g}^{(6)}_{\beta,k}}{5!}\hat{\bar{\Phi}}^3_{\beta,k} + \frac{6\bar{g}^{(8)}_{\beta,k}}{7!}\hat{\bar{\Phi}}^5_{\beta,k} + \cdots}, \tag{3.22}$$

which allows us to explore the variation of $\hat{\bar{\Phi}}_{\beta,k}$ with k. While the flow of the theory can be monitored near the origin $\bar{\Phi}=0$ by defining the vertex functions there, one may also track the evolution of the theory defined at the scale-dependent minimum, i.e., instead of (3.16), we expand the blocked potential as

$$\bar{U}_{\beta,k}(\bar{\Phi}) = \sum_{m=0}^{\infty}\frac{\hat{\bar{g}}^{(m)}_{\beta,k}}{m!}\big(\bar{\Phi}-\hat{\bar{\Phi}}_{\beta,k}\big)^m, \qquad \hat{\bar{g}}^{(m)}_{\beta,k} = \bar{U}^{(m)}_{\beta,k}(\hat{\bar{\Phi}}_{\beta,k}). \tag{3.23}$$

By solving the RG equations explicitly, we illustrate in Fig. 2 the pattern of symmetry restoration above T_c.

IV. DIMENSIONAL CROSSOVER AND THE CRITICAL EXPONENTS

A. Dimensional Crossover

In exploring the critical behavior of our blocked system, with k^{-1} being the characteristic linear dimension for the R^d submanifold, it is desirable to have the coarse-graining procedure repeated until the block length becomes comparable to the correlation length ξ which conventionally is measured by the inverse of the effective mass parameter, i.e., $\xi \sim \mu_\beta^{-1}$. With $k \sim \mu_\beta$, as $\mu_\beta \to 0$ in the neighborhood of the transition where the volume of individual blocks tend to infinity ($k = 0$), the correlation length also becomes infinite. As we have seen before, whether the transition is $d+1$- or d-dimensional is intimately related to the value of $\bar{\beta}$.

In the Introduction, we also argued that beyond the crossover regime, the set of effective degrees of freedom is expected to change. To identify the new degrees of freedom associated with small $\bar{\beta}$ limit, one first makes use of (1.2) to decompose the original unblocked field $\phi(\mathbf{x})$ into the static "light" mode $\phi_0(\mathbf{x})$ and the "heavy" mode $\phi_{n\neq 0}(\mathbf{x})$. In this manner, the classical action in (2.1) can be rewritten as [16]

$$S_\beta[\phi] = \beta \int_{\mathbf{x}} \Big[\frac{Z}{2}(\nabla\phi_0)^2 + V(\phi_0) \Big] + \int_0^\beta d\tau_x \int_{\mathbf{x}} \Big\{ \frac{Z}{2} {\sum_n}' \phi_n \big(\omega_n^2 - \nabla^2\big)\phi_{-n} + \Delta V(\phi_0, \phi_n) \Big\}, \tag{4.1}$$

where the prime notation implies that the summation is over $n \neq 0$ only. For the familiar $\lambda\phi^4$ theory considered here, (4.1) becomes

$$S_\beta[\phi] = \beta \int_{\mathbf{x}} \Bigg\{ \Big[\frac{Z}{2}(\nabla\phi_0)^2 + \frac{\mu^2}{2}\phi_0^2 + \frac{\lambda}{4!}\phi_0^4 \Big] + \frac{1}{2} {\sum_n}' \phi_n \Big(Z\omega_n^2 - Z\nabla^2 + \mu^2 + \frac{\lambda}{2}\phi_0^2 \Big)\phi_{-n}$$
$$+ \frac{\lambda}{6}\phi_0 {\sum_{n_1,n_2,n_3}}' \phi_{n_1}\phi_{n_2}\phi_{n_3}\delta(n_1+n_2+n_3) + \frac{\lambda}{4!} {\sum_{n_1,\cdots,n_4}}' \phi_{n_1}\cdots\phi_{n_4}\delta(n_1+n_2+n_3+n_4) \Bigg\}. \tag{4.2}$$

Clearly in the limit $\beta \to 0$, the heavy modes are strongly damped in the Boltzmann sum and become decoupled, leaving the static light mode $\phi_0(\mathbf{x})$ as the new effective degree of freedom for the reduced d-dimensional system described by $S_{k,d}[\phi_0] = \beta^{-1} S_\beta[\phi]$ [17]. Even though such decoupling strictly speaking holds only up to $O(\bar{\mu}_\beta^2)$ [18], the dimensionally reduced prescription has been demonstrated to be a good approximation in the small $\bar{\mu}_\beta$ limit [6]. While contributions from the $n \neq 0$ Matsubara modes can be handled perturbatively, the static processes described by the $n = 0$ sector must be treated by non-perturbative techniques. This justifies our use of the smearing function (1.4) which defines the static effective blocked fields $\Phi(\mathbf{x})$ shown in (1.3).

In the extreme limits $\bar{\beta} \gg 1$ and $\bar{\beta} \ll 1$ the critical behavior of the system exhibit $(d+1)-$ and d-dimensional characteristics, respectively. What about the intermediate

values of $\bar{\beta}$? Do they correspond to any *physical* non-integer dimensionality? The answer is negative since in principle one cannot associate $S^1 \times R^d$ with some effective manifold $R^{d_{\rm eff}}$. What we shall find is that in the intermediate range, finite-size effect can change the values of the exponents significantly.

B. Phase Transition and Effective Critical Exponents

For definiteness we take $d = 3$ or $S^1 \times R^3$. We start off by making the thickness of the bulk system infinite, i.e., $\beta = \infty$ and gradually decrease its value. In addition, we assume that the system is initially in the broken phase with a negative renormalized parameter $\tilde{\mu}_R^2 < 0$ at $T = 0$. As T is raised, in the limit $k \to 0$ one expects a second-order phase transition at T_c where the symmetry of the system is restored.

The volume of a real system, however, can never be infinite; the Monte Carlo RG simulations, too, are performed on finite lattices. When dealing with such finite systems, one often employs the finite-size scaling assumption to extract the information associated with the infinite-volume limit. It turns out that with our RG prescription the validity of the finite-size scaling hypothesis can readily be tested by focusing on the behavior of each individual block of volume βk^{-d}. To accomplish this, we note that although phase transitions in a strict sense can occur only in the infinite volume limit, a k-dependent "pseudo-transition" temperature $T_c(k)$ can be defined as the scale at which the minimum as well as the mass parameter vanish continuously *within* each block, i.e.,

$$U'_{\beta_c(k),k}(0) = 0, \qquad \text{and} \qquad U''_{\beta_c(k),k}(0) = 0. \tag{4.3}$$

Our $T_c(k)$ is analogous to the rounding temperature commonly encountered in the Monte Carlo RG simulations. One may justify the use of (4.3) as the working definition of the pseudo-transition by noting that in principle there always exists a $T_c(k)$ at which the effective mass parameter $U''_{\beta_c(k),k}$ vanishes regardless of the size of the block, provided that T be kept substantially below Λ in order to be considered as physical. While the finiteness of the blocks makes the transition temperature necessarily k-dependent, the true critical temperature is given by $T_c = T_c(k = 0)$ which is independent of k. We emphasize that our system is always infinitely large with an infinite number of blocks which represent the effective degrees of freedom. However, what we are primarily interested here is to elucidate the critical behavior of the whole system by looking into just one single block whose size varies between zero ($\beta\Lambda^{-3}$) and infinity. The measured critical exponents will exhibit three- and four-dimensonal characteristics for $T_c(k = 0)$ and $T_c(k = \Lambda)$, respectively, with the latter being the original "unblocked", bare system.

There exists two separate methods for investigating the "critical" behavior of a given block. The first one is to fix the theory at $T_c(k)$ and examine the variation of the thermodynamical quantities with T in this vicinity. In the second approach, one first determine the true T_c by taking the $k \to 0$ limit, and then inquire how the thermodynamical quantities deviate with k with the help of finite-size scaling assumption. The schematic diagram of the two methods are given in Fig. 3 below.

(1) pseudo-transition at $T_c(k)$

If we stop the RG flow at some arbitrary scale k, then in the neighborhood of $T_c(k)$, the *effective* critical exponents can be related to the thermodynamical quantities in the following manner:

$$
\begin{aligned}
&\chi^{-1} = \mu^2_{\beta,k} \sim |T - T_c(k)|^{\gamma_{\rm eff}}, && \beta \to \beta_c(k),\\
&\eta_{\rm eff} = -\frac{\partial\ \ln\mathcal{Z}_{\beta_c,k}(\hat\Phi_{\beta_c,k})}{\partial \ln k}, && \hat\Phi_{\beta_c,k} \to 0,\\
&\hat\Phi_{\beta,k} \sim |T - T_c(k)|^{\beta_{\rm eff}}, && \beta \to \beta_c(k),\\
&\hat\Phi_{\beta_c(k),k} \sim h^{1/\delta_{\rm eff}}, && h \to 0.
\end{aligned}
\tag{4.4}
$$

Notice that while the susceptibility diverges at $T_c(k)$, the correlation length given by $\xi \sim (\mu^2_{\beta_c(k),k} + k^2)^{-1/2} \sim k^{-1}$ remains finite as long as $k \neq 0$. This also implies that the conventional expression $\xi \sim |T - T_c(k)|^{-\nu_{\rm eff}}$ is not directly applicable when dealing with a finite system whose correlation length cannot diverge. In the above, the exponent $\delta_{\rm eff}$ gives a measure of how the "magnetization" $\hat\Phi_{\beta,k}$ varies with an external field h as $h \to 0$ at $T_c(k)$. Its measurement is achieved by coupling to Φ a constant source term h.

The effective exponents obtained in this manner will certainly depend on $\bar\beta$, e.g., $\gamma_{\rm eff} = \gamma_{\rm eff}(\bar\beta)$, and the dependence is lifted only when $\bar\beta = 0$ or $\bar\beta = \infty$. In fact, we shall see that $\gamma_{\rm eff}(0) = \gamma_3$ and $\gamma_{\rm eff}(\infty) = \gamma_4$, where the subscripts correspond to the physical dimensionality D. To what extent the system will deviate from its true critical behavior can be analyzed from the relative shift of the pseudo-critical temperature due to the finiteness of the block:

$$
\lim_{k\to 0} |T_c - T_c(k)| \sim k^{\theta}. \tag{4.5}
$$

Numerically, we find $\theta = 1.49 \pm .01$ using the polynomial expansion of $U_{\beta,k}(\Phi)$ up to $O(\Phi^8)$. The value of θ in general can be related to $\nu_{\rm eff}$ as $\theta = \nu^{-1}_{\rm eff}(0) = \nu_3^{-1}$. Notice that one may also take the opposite limit $k \to \Lambda$. In this case, $\theta \approx 2$, leading to $\nu_4 = 1/2$.

(2) finite-size scaling:

Based on the principle of finite-size scaling, one may assume that near T_c thermodynamical quantities depend only on the dimensionless ratio ℓ/ξ where ℓ and ξ are, respectively, the characteritic finite size and the correlation length of the system. For example, the susceptibility χ can be written as

$$
\chi = \xi^{\gamma_{\rm eff}/\nu_{\rm eff}} f\big(\frac{\ell}{\xi}\big), \tag{4.6}
$$

where $f(r)$ is a scaling function with the property $f(r) \to$ const. as $r \to \infty$. With the presence of two finite length scales, $\ell = \beta$ and k^{-1}, in our system, two different scalings are expected. For $\ell = k^{-1}$, using the relation $\xi^{1/\nu_{\rm eff}} \sim |T - T_c(k)|^{-1}$, (4.6) can then be rewritten as

$$
\chi = |T - T_c(k)|^{-\gamma_{\rm eff}} f\big(\frac{|T - T_c(k)|^{-\nu_{\rm eff}}}{k}\big), \tag{4.7}
$$

which shows that f depends on the scaling variable $Y_\beta = |T - T_c(k)|/k^{1/\nu_{\rm eff}}$. This "$Y_\beta$" scaling prescription which corresponds to the pseudo-transition we have just addressed, explores the T dependence near $T_c(k)$ while keeping k fixed. The pattern of crossover associated with Y_β is depicted in Fig. 4.

On the other hand, if we choose $\ell = \beta_c$, the k-independent inverse critical temperature and let the correlation length of the finite system be given by the maximum linear dimension of the block, i.e., $\xi \sim k^{-1}$, (4.6) then takes on the form

$$\chi = k^{-\gamma_{\rm eff}/\nu_{\rm eff}} \tilde{f}\big(\beta_c k\big) = k^{-\gamma_{\rm eff}/\nu_{\rm eff}} \tilde{f}\big(\bar{\beta}_c\big), \tag{4.8}$$

where $\tilde{f}$ is another scaling function with $Y_k = \bar{\beta}_c$ being the scaling variable. This is completely consistent with our claim that $\bar{\beta}$ sets the scale of dimensional crossover. Thus, instead of extracting the exponents from the T dependence of the thermal parameters near $T_c(k)$, we may go directly to the true T_c and explore the k dependence of thermodynamical quantities, viz,

$$\chi^{-1} \sim k^{\gamma_{\rm eff}/\nu_{\rm eff}}, \qquad\qquad \hat{\Phi}_{\beta_c,k} \sim k^{\beta_{\rm eff}/\nu_{\rm eff}}. \tag{4.9}$$

The pattern of crossover associated with Y_k is exemplified in Fig. 5.

It is important to note that although thermodynamical quantities may diverge or vanish at $T_c(k)$, they remain finite at the true critical temperature T_c for $k \neq 0$, as can be seen from Fig. 6 for the minimum $\hat{\Phi}_{\beta,k}$. Comparing the two procedures, we see that while examining the T dependence of the physical quantities near $T_c(k)$ allows us to extract the critical exponents, exploring the k dependence at T_c in the second approach yields exponents as ratios of $\nu_{\rm eff}$ which must be extracted from

$$\lim_{k\to 0}\ \lim_{T\to T_c} |T^2 - T_c^2(k)| \sim k^{1/\nu_{\rm eff}(k)}. \tag{4.10}$$

Comparison with (4.5) implies $\theta = \nu_{\rm eff}^{-1}(0)$. Various ratios of the exponents extracted by polynomial truncation without the inclusion of $\mathcal{Z}_{\beta,k}$ are summarized in Table 2. In addition, since $\lambda_{\beta,k}/\bar{\mu}_{\beta,k}$ approaches a constant as $k \to 0$ and $T \to T_c$ and $\lambda_{\beta_c,k=0} = 0$ [6] [19], we see that both $\lambda_{\beta,k}$ and $\mu_{\beta,k}$ must vanish at the same rate, i.e.,

$$\lambda_{\beta,k=0} \sim |T - T_c|^{\zeta_{\rm eff}(0)} \sim |T - T_c|^{\nu_{\rm eff}(0)}, \tag{4.11}$$

which allows for the identification $\zeta_{\rm eff}(0) = \nu_{\rm eff}(0)$ when $\eta_{\rm eff} = 0$. In other words, neglecting the effect of $\mathcal{Z}_{\beta,k}$, $\nu_{\rm eff}(0)$ can be deduced by examining how λ_β vanishes as T_c is approached.

V. NUMERICAL METHODS AND RESULTS

We now describe the details pf how the differential RG equations (3.3) and (3.4) are solved numerically. The evolution begins at the UV cutoff scale Λ where all thermal effects vanish and the running parameters take on the bare values which we denote with the subscript B. We choose $\mu_B^2 < 0$ and a small positive value for the coupling constant λ_B.

The symmetric phase is characterized by a positive value of the renormalized mass $\tilde{\mu}_R^2 > 0$ even if the initial value $\mu_B^2 < 0$. On the other hand, if $\tilde{\mu}_R^2 < 0$ the theory is in the spontaneously broken phase. In Fig. 7 we depict the dependence of the minimum, mass, and coupling constant near $T_c(k=0)$.

As noted before, phase transitions strictly speaking can only take place at $k = 0$ where the block volume becomes infinite; however, to map out the entire crossover regime, one must study the dependence of $U_{\beta,k}(\Phi)$ on $\bar{\beta}_c$. To accomplish this we integrate the *dimensionful* versions of (3.3) and (3.4) as well as the truncated flow equations such as (3.17) - (3.19) down to a small value $k \neq 0$ for which the pseudo-transition temperature is $T_c(k)$. The value of k can then be adjusted in order to probe different values of $\bar{\beta}_c$. Decreasing k corresponds to enlarging the size of the blocks. The gradual change of the effective exponents with $\bar{\beta}$ is illustrated in Fig. 4 where the different symbols correspond to different values of k.

From Fig. 4 (a), we see that true scaling takes place only near integer dimensions where k is either very large or vanishing. In these neighborhoods, the measured values of the effective exponents are approximately constant for a large range of $T - T_c(k)$ with only minute deviations. However, in the intermediate regime, complete scaling does not occur and the effective exponents can only be approximated by fixing the value of k and selecting a narrow range of $|T - T_c(k)|$. On the other hand, when parameterized by the finite-size scaling variable Y_β, the data collapse nicely into one curve of $\gamma_{\rm eff}$ which smoothly interpolates between γ_4 and γ_3, as can be seen from Fig. 4(b).

The numerical results indicate that the smaller the value of $\bar{\beta}$, the closer the system is to the three-dimensional critical behavior. On the other hand, a large value of $\bar{\beta}$ implies smaller blocks and the exponents will crossover to the four-dimensional values. This can be explained by observing that as the individual blocks shrink, the system begins to flow back to the original four-dimensional unblocked system and the bare theory is recovered when the linear dimension of the block becomes Λ^{-1}. Since thermal fluctuations are highly suppressed at $k \sim \Lambda$, quantum fluctuations become dominant and "recreate" the extra dimension, making the effective dimensionality four..

The truncated flow equations were integrated using a fifth-order adaptive step-size Runge-Kutta integrator allowing for precise determination of the thermal parameters near $T_c(k)$. In Fig. 5 the $\bar{\beta}$ dependence of the critical exponent $\gamma_{\rm eff}$ calculated from an eighth order polynomial truncation of our flow equations is illustrated. The exponents are independent of the values of the bare mass parameter, and coupling constant, as required by universality. By comparing the results obtained with and without truncation, we see that the latter yields a better agreement with the experimental values. While the crossover scale is expected to be $Y_k^{1/\nu_{\rm eff}} = \bar{\beta}_c = 1$, the truncated RG equation shows that the three-dimensional characteristics of the theory is uncovered only when $\bar{\beta}_c$ is substantially less than unity ($\approx 10^{-6}$ from the Fig. 5). Increasing the order of truncation only affects the result slightly. Although qualitatively correct, this may be an indication of the inadequacy of the polynomial approximation in exploring the nonperturbative phenomenon of dimensional crossover.

In both Figs. 4 and 5 the values of the critical exponents calculated by explicitly integrating over (3.3) and (3.4) without polynomial truncation are also given. In carrying out the calculation, $U_{\beta,k}(\Phi)$ was discretized with $\Delta\Phi = 0.01$ and the cutoff was lowered

systematically from $k = \Lambda$ to $k = 0$ using finite differences. The k steps were second-order Runge-Kutta and the Φ derivatives were calculated using a five-point difference. Unfortunately, due to the amount of computer time required for making each run, we were unable to produce plots of the effective exponents as a function of the continuous variable $\bar{\beta}_c$ from the nontruncated RG equations. Instead, the dependence of the effective exponents on $\bar{\beta}$ which takes on continuous values is reported by using the truncated flow equations obtained in Sec. III.

We list in Table 1 the values of the critical exponents obtained from various orders of polynomial truncation of (3.16) along with those calculated by solving (3.3) and (3.4) directly without truncation. As mentioned before, the three- and the four-dimensional limits are reached by dialing the value of $\bar{\beta}$ to be such that $\bar{\beta} \ll 1$ and $\bar{\beta} \gg 1$, respectively. ¿From the Table, one sees that $\mathcal{Z}_{\beta,k}$ only gives a minute corrections to the exponents. Nevertheless, its consideration is crucial for extracting the anomalous dimension η_3. The results agree very well with that derived from ϵ expansion up to ϵ^5 and five loops [20]. Note that presently we were only able to extract $\nu_{\rm eff}$ indirectly from studying the T dependence of $\lambda_{\beta,k=0}$ in (4.11) using truncated polynomial expansion. The measurement of $\alpha_{\rm eff}$ turns out to be rather difficult. We also comment that extracting the critical exponents $\beta_{\rm eff}$, $\eta_{\rm eff}$, and $\delta_{\rm eff}$ using the truncated flow equations requires an expansion about the k-dependent minimum $\hat{\Phi}_{\beta,k}$ of the blocked potential $U_{\beta,k}(\Phi)$. However, we find that the polynomial expansion around $\hat{\Phi}_{\beta,k}$ given by (3.23) fails near T_c even though away from T_c it agrees well with that obtained using (3.16). This is due to the nonanalytic behavior of $U_{\beta_c,k}(\Phi)$ as $k \rightarrow 0$.

In Table 2 we show the k dependence of the $(2m)$-point vertex functions $g^{(2m)}_{\beta,k}$ up to $m = 5$. From the Table, we see that the vertex functions diverge in the limit $k \rightarrow 0$ for $m \geq 4$, and the diverging behaviors ($\sim k^{3-m}$) are in accord with (3.5) where one writes $g^{(2m)}_{\beta,k} = \beta^{-1+m} k^{3-m(1+\eta_{\rm eff})} \bar{g}^{(2m)}_{\beta,k}$ for $d = 3$, apart from $\eta_{\rm eff}$. According to the Table, $g^{(6)}_{\beta_c,k}$ approaches a constant as $k \rightarrow 0$; however, by taking into consideration the anomalous dimension, one would find that it diverges as $k^{-3\eta_{\rm eff}}$. Notice that the values $a_2 = 1.998 \pm 0.005$ and $a_4 = 0.997 \pm 0.004$ measure, respectively, the ratios of the effective critical exponents $\gamma_{\rm eff}/\nu_{\rm eff}$ and $\zeta_{\rm eff}/\nu_{\rm eff}$. The fact that $a_4 = \zeta_{\rm eff}/\nu_{\rm eff}$ is close to unity gives support to our previous claim that $\nu_{\rm eff}$ can be approximated by $\zeta_{\rm eff}$. We also see that the manner in which the vacuum expectation value $\hat{\Phi}_{\beta_c,k}$ vanishes as function of k allows for the identification $\beta_{\rm eff}/\nu_{\rm eff} = 0.51 \pm 0.01$. After extracting $\zeta_{\rm eff} = \nu_{\rm eff}$ by measuring the k dependence of $|T - T_c(k)|$ as $k \rightarrow 0$ using (4.10), all other exponents can be deduced from finite-size data. The procedure is analogous to the standard approach employed in the Monte Carlo simulations of finite-size systems. Using $\nu^{-1}_{\rm eff}(0) = \nu^{-1}_3 = 1.49 \pm 0.01$, we see that the exponents extracted from Table 2 are in good agreement with those measured by examining the dependence of the thermal parameters near $T_c(k)$.

The numerical values we obtained using polynomial truncation in $U_{\beta,k}(\Phi)$ are in excellent agreement with [21] and [15]. However, as demonstrated by Morris who incorporated polynomial contributions up to $O(\Phi^{50})$ [15], the critical exponents derived in this manner do not uniformly converge to the experimental values but instead exhibit oscillatory behavior. Although the oscillation has not yet been observed in our truncated scheme which includes contributions only up to Φ^{10}, we remark that this generic feature is another indication of the breakdown of the polynomial expansion of $U_{\beta,k}(\Phi)$ in (3.16).

The presence of the nonanalytic structure in the blocked potential can be seen readily by noting that in the zero-temperature "dressed" ($V_R''(\Phi) \to V_k''(\Phi)$) approach, the potential contains a term of the form $(k^2 + V_k''(\Phi))^2 \ln(1 + V_k''(\Phi)/k^2)$, where $V_k''(\Phi) = \mu_k^2 + \lambda_k \Phi^2/2$ [9]. When $V_k''(\Phi)/k^2$ is small, the logarithmic term may be expanded as $\ln(1+V_k''(\Phi)/k^2) = \sum_{m=1}^{\infty}(-1)^{(m+1)}(V_k''(\Phi)/k^2)^m/m$ which allows us to parameterize the blocked potential as polynomials of Φ. While such expansion is justified in the large k limit, it generally breaks down in the IR limit where $k \to 0$ even though λ_k may be small. In particular, when V_k'' vanishes slower than k^2, the logarithmic dependence of Φ can no longer be expanded as power series. If one insists on making the expansion, the resulting series will be divergent with alternating sign in λ_k. Similar arguments also hold for finite-temperature systems. We emphasize that the IR singularities encountered in $g_{\beta,k}^{(2m)}$ for $m \geq 4$ illustrated in Table 2 are merely an artifact which arises from the breakdown of polynomial expansion of $U_{\beta,k}(\Phi)$ close to T_c and $k = 0$ where both the mass parameter $\mu^2_{\beta_c,k=0}$ and the coupling strength $\lambda_{\beta_c,k=0}$ vanish. Thus, we conclude that both the presence of IR divergences and oscillation of the critical exponents around experimental values are all due to the breakdown of polynomial expansion close to T_c in the IR regime.

We find further evidence for this claim by examining the Φ dependence of the derivatives of the blocked potential close to T_c. In Fig.89, we plot the fourth derivative of the blocked potential, $U^{(4)}_{\beta_c,k}(\Phi)$, solving the nontruncated RG equations (3.3) and (3.4). Had a fourth order truncation of the blocked potential been sufficient, i.e., $U_{\beta_c,k}(\Phi) = \mu^2_{\beta_c,k}\Phi^2/2 + \lambda_{\beta_c,k}\Phi^4/4! + O(\Phi^6)$, one then will have $U^{(4)}_{\beta_c,k}(\Phi) = \lambda_{\beta_c,k}$ which for any given k is a constant independent of Φ. While the prediction made by polynomial truncation is reasonable for large k, it is clearly unreliable in the small k limit. As depicted in Fig. 8, in the IR regime $U^{(4)}_{\beta_c,k}(\Phi)$ is a complicated nonanalytic function of Φ and not a constant! It turns out that a large number of terms are required if one wishes to characterize the Φ dependence of $U^{(4)}_{\beta,k}(\Phi)$ by powers series. For example, the inclusion of the sixth-order term would yield a parabolic shape to the $U^{(4)}_{\beta_c,k}(\Phi)$ near $\Phi = 0$ and other higher-order contributions with increasing magnitude and oscillating signs must be included to fit the plateau at large Φ [4]. That is, a simple polynomial fit seems feasible for sufficiently small Φ, but becomes unnatural in the plateau region. The difficulty encountered in fitting $U^{(4)}_{\beta_c,k}(\Phi)$ with power series of Φ casts doubts on the use of polynomial expansion of $U_{\beta,k}(\Phi)$ in the limit $k- > 0$.

VI. SUMMARY AND DISCUSSIONS

We have demonstrated in this paper the applicability of RG in describing dimensional crossover. The nonperturbative nature of the RG formulation allows us to establish a smooth connection between the physics in $d+1$ and d dimensions. The scale at which crossover takes place is shown to be at $\bar{\beta} \sim 1$. Through numerical integration of the coupled RG equations in (3.3) and (3.4), the dependence of the effective critical exponents on $\bar{\beta}$ was explored. We find that in the limits $\bar{\beta} \gg 1$ and $\bar{\beta} \ll 1$, the transitions can be associated with $(d+1)$ and d dimensions, respectively. In the intermediate values of $\bar{\beta}$, finite-size effects are important.

The effective exponents were extracted using two different methods: In the first approach, we measure the effective critical exponents by studying the temperature dependence of the thermodynamical quantities near $T_c(k)$ with the true critical behavior obtained when $k \to 0$. In the second approach, we fixed the temperature at T_c and examined the k dependence of the thermal parameters. Invoking finite-size scaling arguments, the critical exponents were obtained as ratios of $\nu_{\rm eff}(k)$ which was determined using the relation $\lim_{k\to 0} |T^2 - T_c^2(k)| \sim k^{1/\nu_{\rm eff}(k)}$ given in (4.10). Our measurements of the effective critical exponents for $D = 4$ $(k \to \Lambda)$ and $D = 3$ $(k \to 0)$ are in excellent agreement with the established results, and are more conclusive than those presented in [19] where a smooth regulator was used. In the intermediate range of $\bar{\beta}$ where scaling is only approximate, the exponents are deduced by fixing the value of k and restricting the range of $|T - T_c(k)|$. The numerical results derived from both schemes are completely consistent and allow for a smooth interpolation of the effective exponents between the three- and four-dimensional values.

The success of our formalism in exploring the physics of crossover is attributed to its accurate tracking of the effective degrees of freedom at any arbitrary point in the two-parameter space spanned by the flow parameters β and k. Our approach allows us to resum a large class of diagrams that are beyond the five-loop calculations up to ϵ^5 carried out in [20]. In addition, it is computationally far more advantageous to solve the differential RG equations (3.3) and (3.4) than to work out hundreds of Feynman graphs to achieve the same level of accuracy. While our RG equations take into consideration all possible nonoverlapping Feynman graphs to infinite loop orders, the ϵ expansion technique provides individual treatments to distinct Feynman graphs, both overlapping and nonoverlapping, only to a finite loop order. It remains an interesting issue to explore the role of the overlapping graphs that are left out in our RG prescription. Nevertheless, we believe that our formalism is superior to the ϵ expansion since the critical exponents deduced from the former are manifestly finite and agree remarkably well with experiments; as for the latter, it is in reality an asymptotic series in which truncations must be made in order to attain agreement with the experimental measurements.

In light of the success of our RG prescription in the investigations of dimensional crossover as well as critical behavior of the scalar $\lambda\phi^4$ model, one can readily extend the formalism to $O(N)$ symmetry in order to study the XY $(N = 2)$ model, Heisenberg $(N = 3)$ model, and the large N limit [22]. Systems at lower dimensionality can also be studied since the validity and generality of our treatment can be tested by comparing the results with the exact solutions possible in certain two-dimensional systems. For the XY model, spontaneous symmetry breaking can take place for $D > 2$ at finite T_c but at exactly $D = 2$ the system exhibits the well known Kosterlitz-Thouless phase transition driven by vortices which are the topologically singular spin configurations. It would be interesting to see if our formalism can capture the topological crossover as the effective dimension changes. Another possible application is the consideration of the $3 + 1$ $U(1)$ scalar QED which has same critical behavior as the three-dimensional superconductor. Since the effects of quantum fluctuations are also incorporated in a systematic manner, one may also consider quantum critical phenomena as in [23]. The RG formalism is not just limited to scalar theories, but can be implemented in fermionic systems in a similar fashion, as has been carried out recently in [24], and in gauge theories [25] [26]. These issues shall be addressed in the future publications.

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FIGURE CAPTIONS

Figure 1. Temperature dependence of the wavefunction renormalization constant $\mathcal{Z}_{\beta,k=0}(\Phi)$. Inset shows the Φ dependence of $\mathcal{Z}_{\beta,k=0}(\Phi)$ at $T = 0$.

Figure 2. Blocked potential $U_{\beta,k}(\Phi)$ near T_c. The transition is second order.

Figure 3. Schematic diagram of the two methods used for measuring the critical exponents. The left block summarizes method (1) and the right method (2).

Figure 4. Effective critical exponent $\gamma_{\rm eff}$ as function of (a) $\log_{10}(T - T_c(k))$ and (b) $\log_{10}((T - T_c(k))/k^{1/\nu_{\rm eff}})$, where the three-dimensional value $\nu_3^{-1} = \nu_{\rm eff}^{-1}(0) = 1.49 \pm 0.01$ is obtained via measuring the shift $|T_c - T_c(k)|$ due to the finiteness of k. The different symbols correspond to different values of k.

Figure 5. Dependence of $\gamma_{\rm eff}$ on $\bar{\beta}_c$ obtained via polynomial expansion of $U_{\beta,k}(\Phi)$ up to $O(\Phi^8)$. The values of γ_3 and γ_4 calculated from integrating the RG improved versions of (3.1) and (3.2) are also included.

Figure 6. Dependence of the minimum , $\hat{\Phi}_{\beta,k}$, on the IR cutoff k at the critical temperature $T = T_c(k = 0)$, showing that $\hat{\Phi}_{\beta,k}$ only vanishes when $k = 0$.

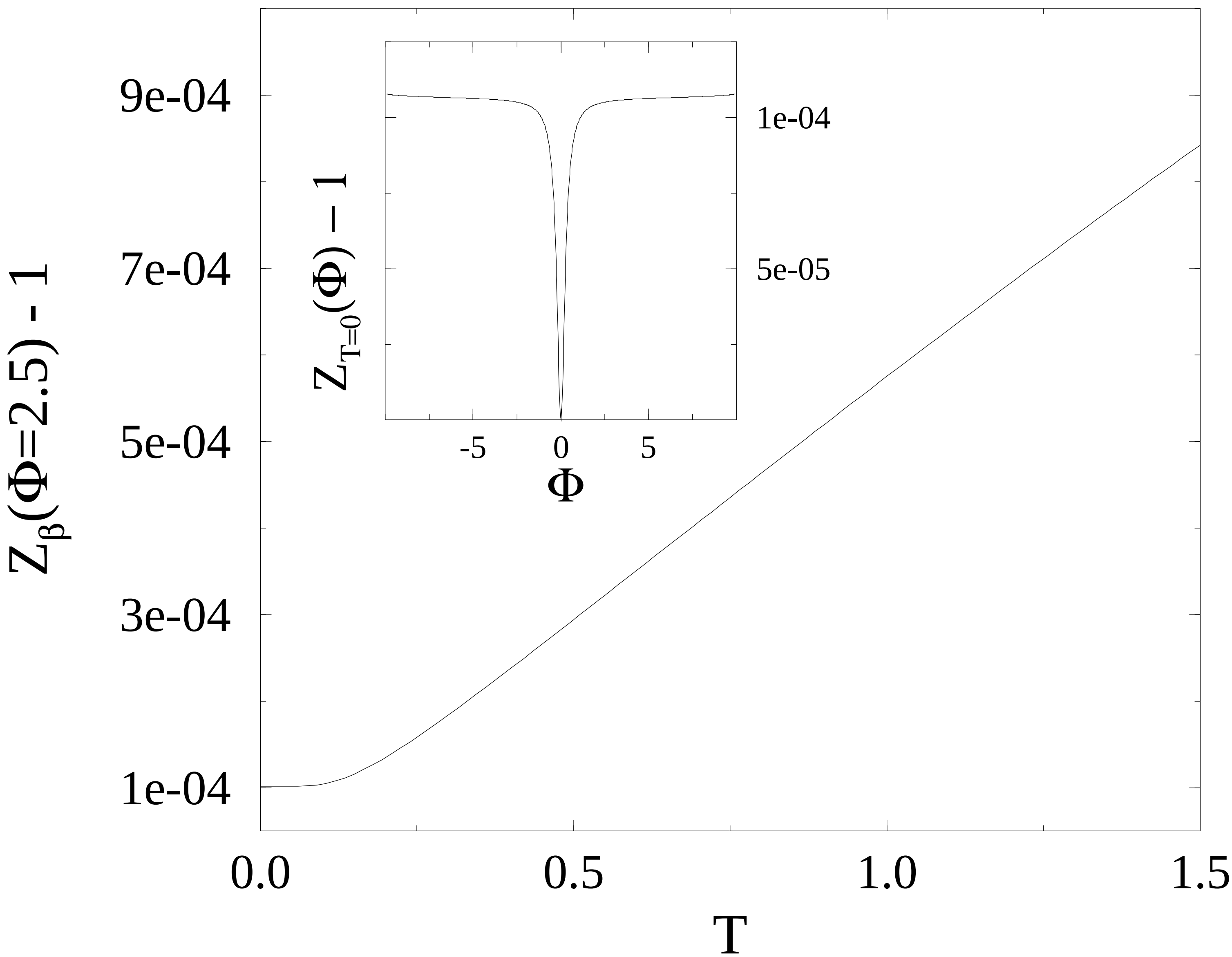
Figure 7. Temperature dependence of the thermal mass parameter μ_β^2, coupling constant λ_β, and the minimum $\hat{\Phi}_{\beta,k}$ near $T = T_c$. The T dependence of λ_β in the symmetric phase is also illustrated.

Figure 8. $U_{\beta,k}^{(4)}$ as a function of Φ close to T_c $(T - T_c(k) \sim 10^{-5})$ calculated with the nontruncated RG equations.

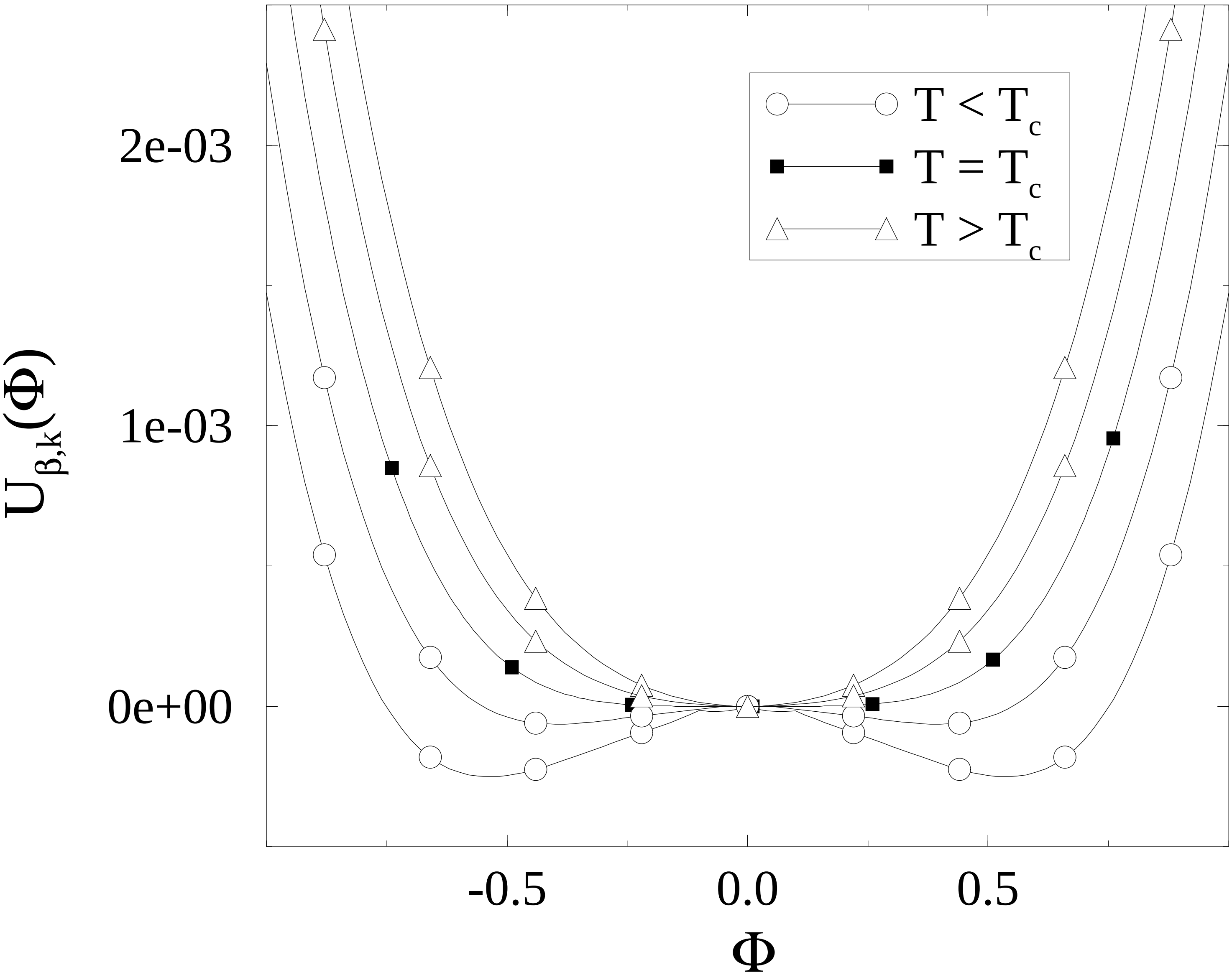
TABLES

Table 1. Critical exponents as function of the level of truncation along with the best calculations to date and experimental values. NT indicates results obtained from the nontruncated RG equations (3.3) and (3.4).

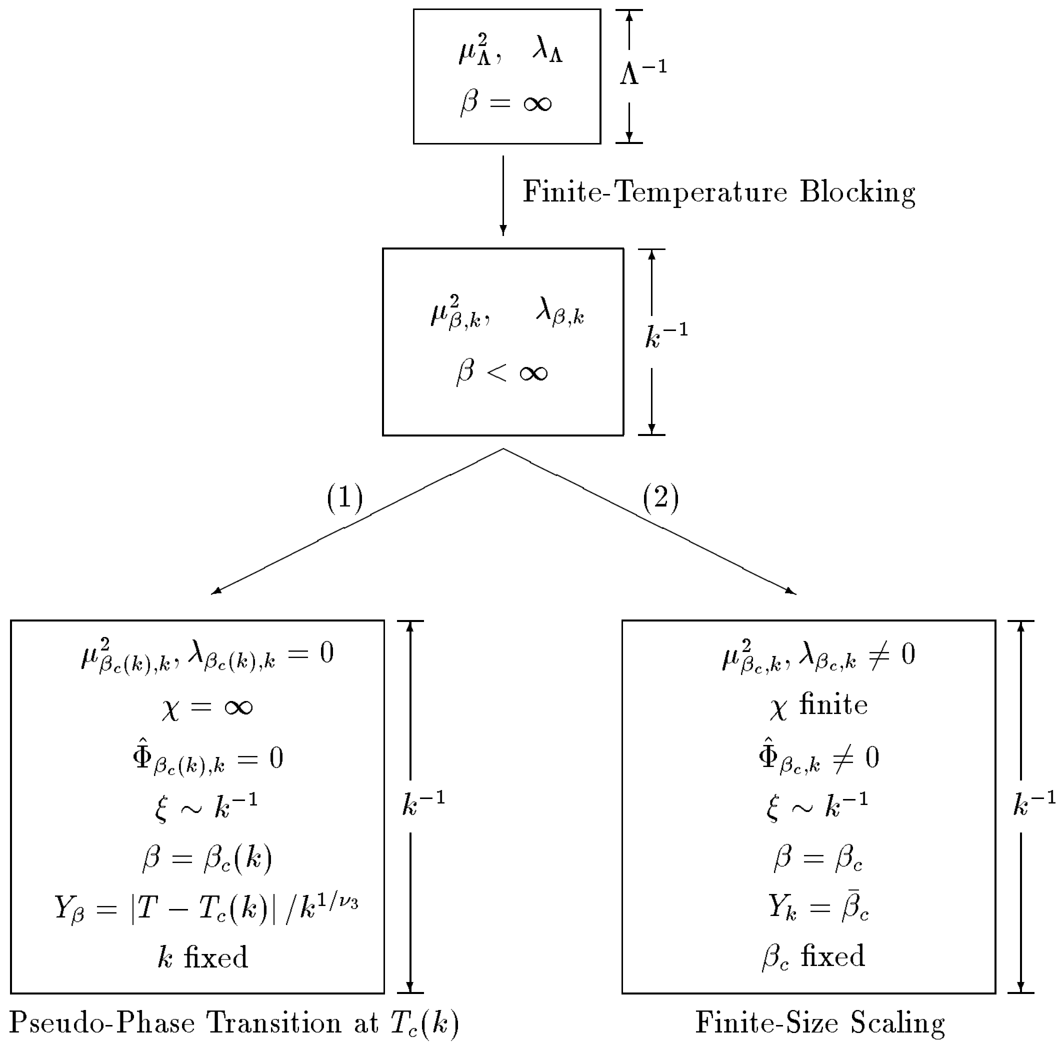
Table 2. k dependence of vertex functions at T_c in the absence of wavefunction renormalization.



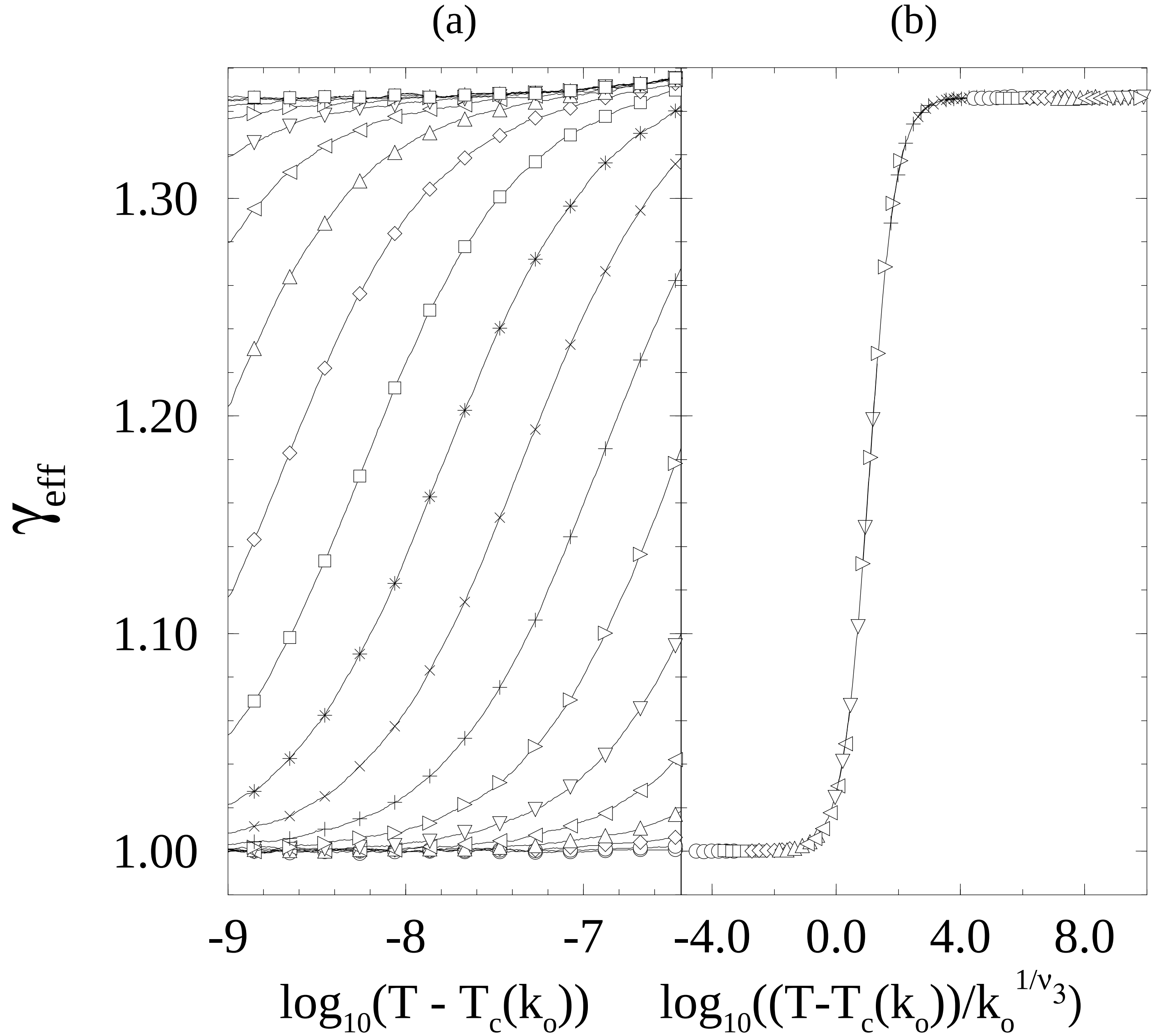
$Z_\beta(\Phi=2.5) - 1$
9e-04
7e-04
5e-04
3e-04
1e-04
0.0
0.5
1.0
1.5
T
$Z_{T=0}(\Phi) - 1$
1e-04
5e-05
-5
0
5
Φ



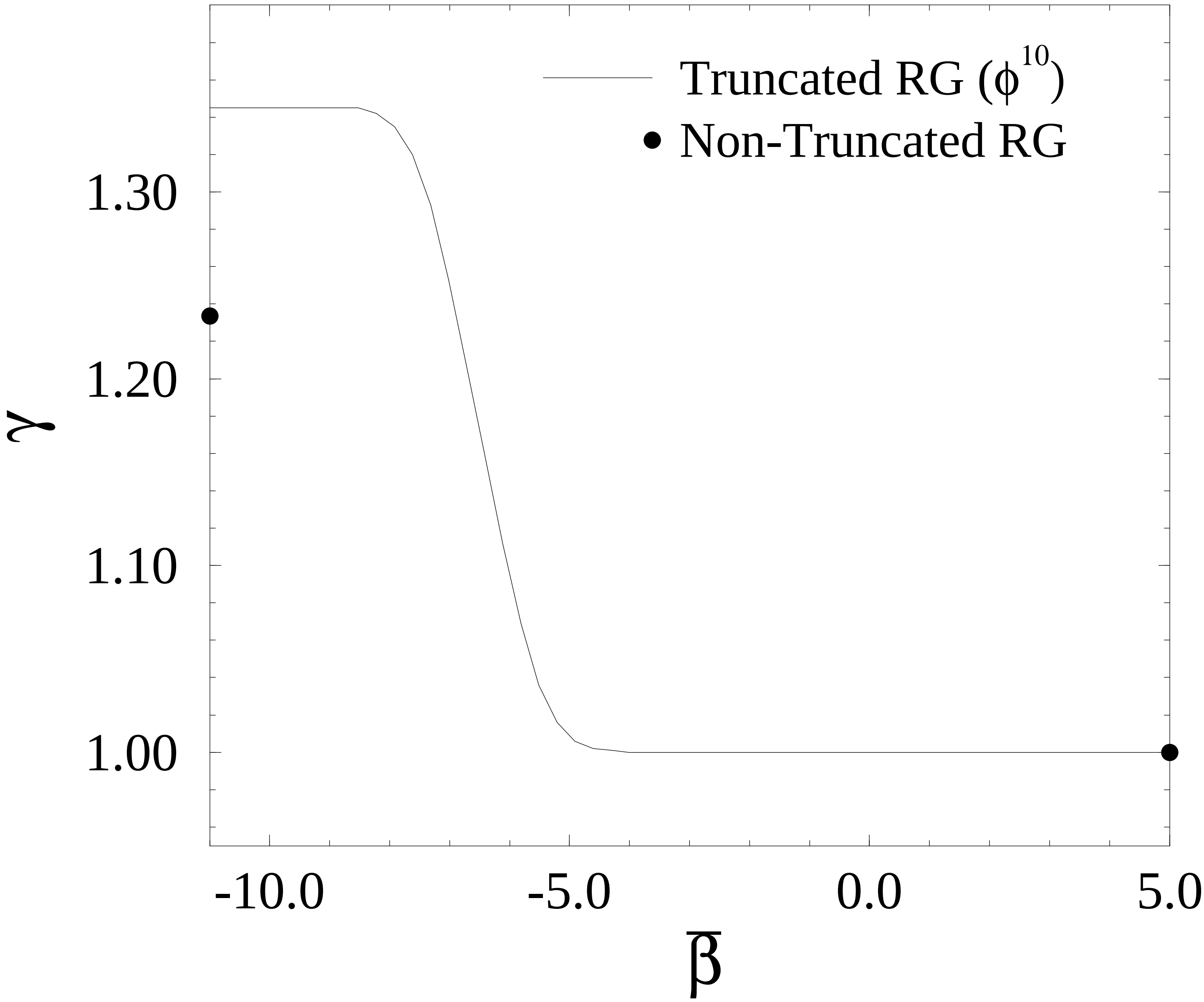
T < Tc
T = Tc
T > Tc
2e-03
1e-03
0e+00
Uβ,k(Φ)
-0.5
0.0
0.5
Φ



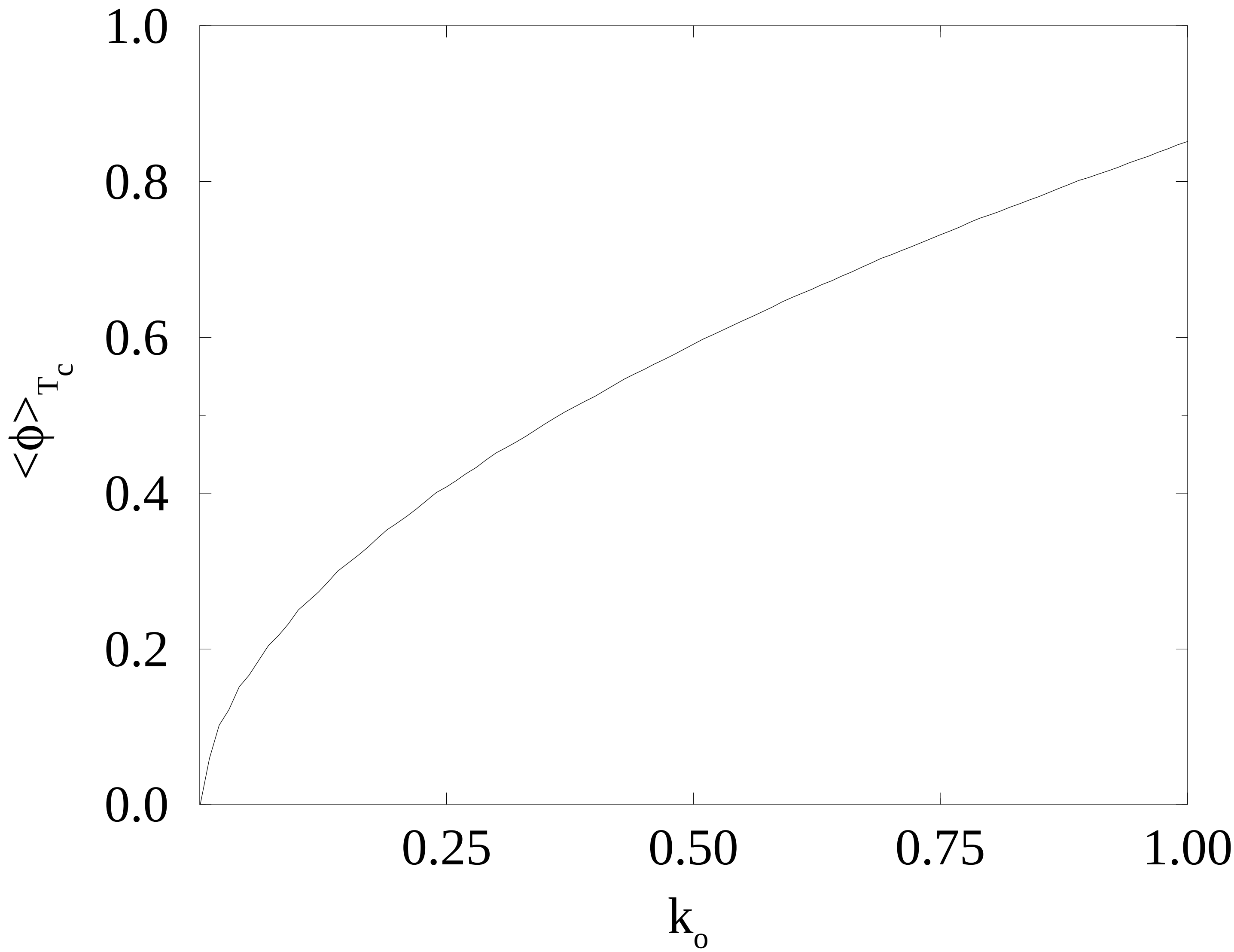
$\mu^2_\Lambda, \quad \lambda_\Lambda$
$\beta = \infty$
Λ^{-1}
Finite-Temperature Blocking
$\mu^2_{\beta,k}, \quad \lambda_{\beta,k}$
$\beta < \infty$
k^{-1}
(1)
(2)
$\mu^2_{\beta_c(k),k}, \lambda_{\beta_c(k),k} = 0$
$\chi = \infty$
$\hat{\Phi}_{\beta_c(k),k} = 0$
$\xi \sim k^{-1}$
$\beta = \beta_c(k)$
$Y_\beta = \left|T - T_c(k)\right| / k^{1/\nu_3}$
k fixed
k^{-1}
$\mu^2_{\beta_c,k}, \lambda_{\beta_c,k} \neq 0$
χ finite
$\hat{\Phi}_{\beta_c,k} \neq 0$
$\xi \sim k^{-1}$
$\beta = \beta_c$
$Y_k = \bar{\beta}_c$
β_c fixed
k^{-1}
Pseudo-Phase Transition at $T_c(k)$
Finite-Size Scaling

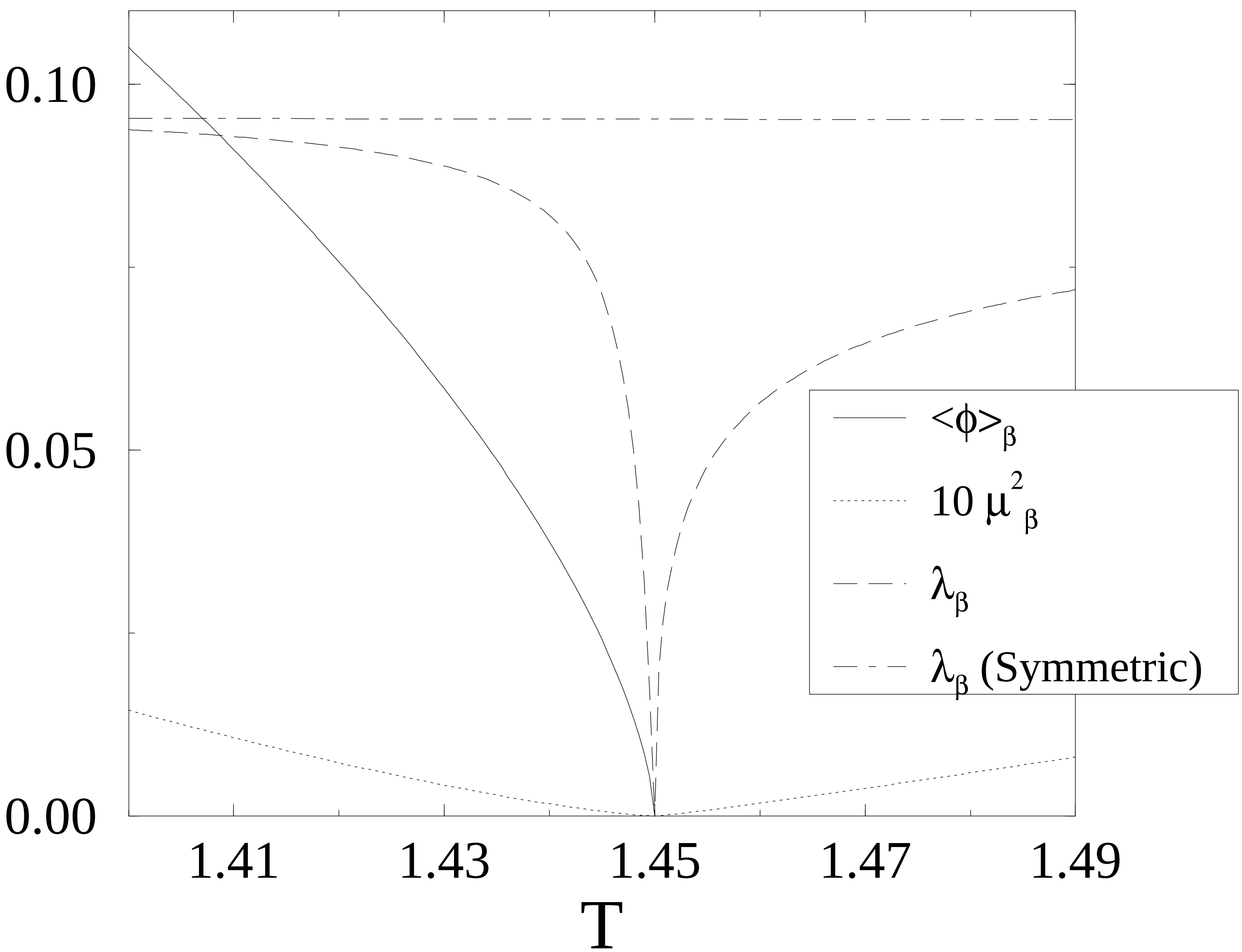


(a)
(b)
1.30
1.20
1.10
1.00
γ_{eff}
-9
-8
-7
-4.0
0.0
4.0
8.0
$\log_{10}(T - T_c(k_o))$
$\log_{10}((T-T_c(k_o))/k_o^{1/\nu_3})$

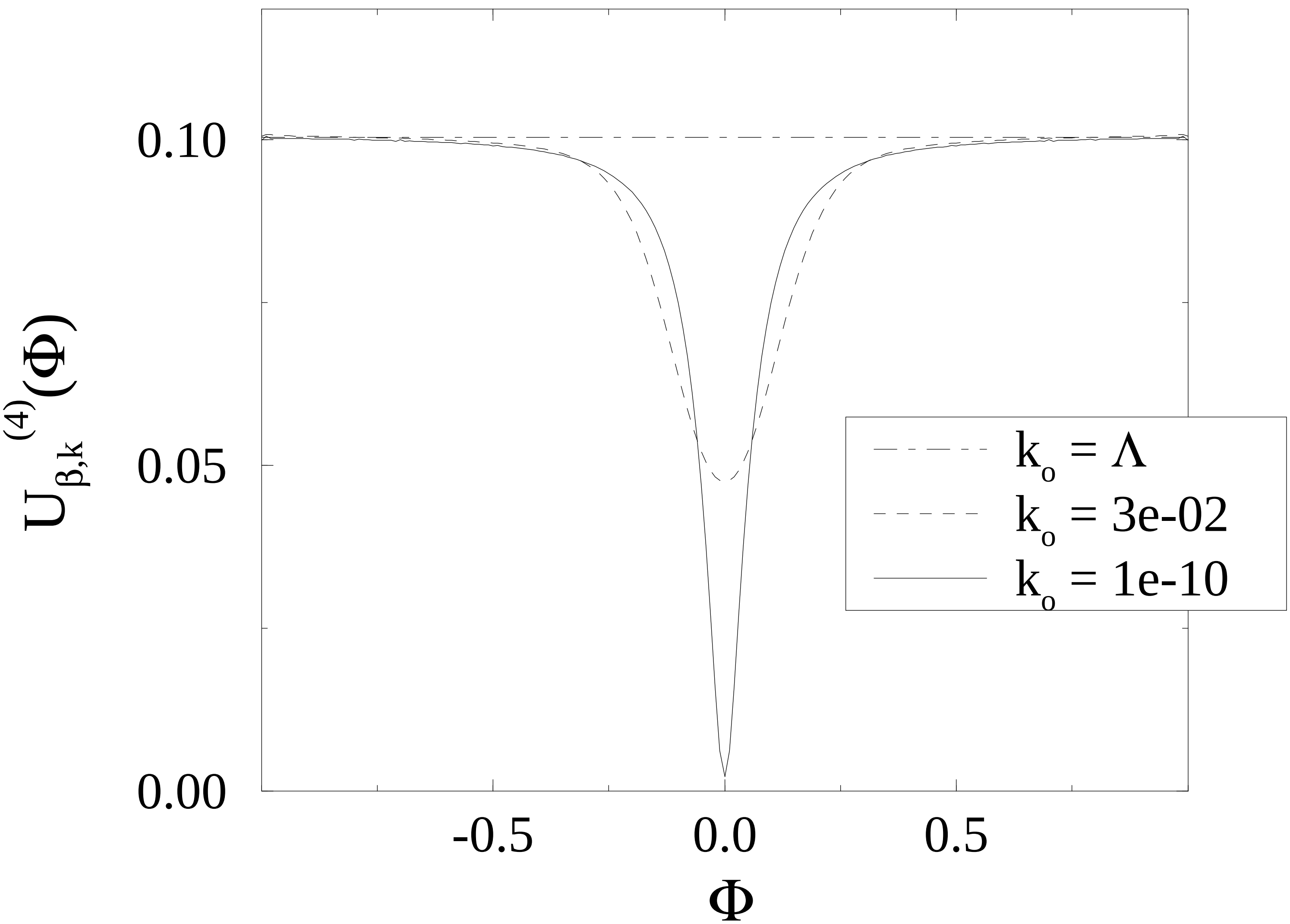


Truncated RG (ϕ^{10})
Non-Truncated RG
1.30
1.20
1.10
1.00
γ
-10.0
-5.0
0.0
5.0
$\bar{\beta}$


1.0
0.8
0.6
0.4
0.2
0.0
0.25
0.50
0.75
1.00
$<\phi>_{T_c}$
k_o



0.10
0.05
0.00
1.41
1.43
1.45
1.47
1.49
T
$<\phi>_\beta$
$10\ \mu^2_\beta$
λ_β
λ_β (Symmetric)



0.10
0.05
0.00
-0.5
0.0
0.5
Φ
U_{β,k}^{(4)}(Φ)
k_o = Λ
k_o = 3e-02
k_o = 1e-10